DOI: 10.5281/zenodo.21385703 Volume 1 | Issue 2 | ISSN: 3141-643X | Published: July 2026

Ktrend - African Journal of Mathematics, Statistics and Computer Science

Strictly Pseudo-Contractive Volterra Integral Operators in Banach Spaces: Weak and Strong Convergence of the Mann Iteration with a Numerical Analysis Extension

I. D. Edem[1,*], J. Akpakwa[2], U. M. Okon[3], W. K. Udogworen[4], O. G. Udoaka[5]

[1,2,3,4,5]Department of Mathematics, Akwa Ibom State University, Nigeria

Corresponding Author: inemesitedem@aksu.edu.ng

Abstract

We study nonlinear Volterra integral equations in uniformly convex Banach spaces $L^p(\Omega)$ $(1 < p < \infty)$ using the language of strictly pseudo-contractive operators defined via the normalized duality mapping J_p. Under a one-sided Lipschitz condition on the kernel (which is strictly weaker than a global Lipschitz condition), we prove that the associated Volterra operator T is k-strictly pseudo-contractive. Using known fixed point theory for such mappings, we establish that the Mann iteration converges weakly to a solution of the integral equation. Strong convergence is obtained under an additional demicompactness assumption, which is verified for a large class of kernels. In the second part, we propose a numerical analysis extension: semi-discretisation in time (method of lines) transforms the Volterra equation into a system of ordinary differential equations. We then apply the Mann iteration as an inexact solver for the resulting nonlinear system and prove convergence of the fully discrete scheme under realistic smoothness assumptions. This provides a rigorous foundation for computational methods. The results are new for Volterra equations with one-sided Lipschitz kernels and open the door to reliable numerical simulations in L^p spaces.

Keywords: Volterra integral equation; strictly pseudo-contractive mapping; Mann iteration; one-sided Lipschitz condition; Banach space; numerical analysis; method of lines.

2020 Mathematics Subject Classification: 45G10, 47H09, 47J25, 65R20.

1. Introduction

Volterra integral equations of the second kind,

$$x(t) = f(t) + \int_0^t K(t, s, x(s))\, ds, \qquad t \in [0, T],$$

arise in numerous applied fields, including population dynamics, viscoelasticity, control theory, and epidemiology [1]. When the kernel K is globally Lipschitz in its third argument, the associated Volterra operator is a contraction on a suitable Banach space (e.g., $C([0, T])$ or $L^p([0, T])$), and the Banach fixed point theorem guarantees existence, uniqueness, and convergence of Picard iterates. However, many physical models involve nonlinearities that are not globally Lipschitz but satisfy a weaker one-sided Lipschitz condition; typical examples are cubic nonlinearities, saturation terms, and certain monotone growth functions. These nonlinearities do not yield a contraction, yet they often preserve enough structure (e.g., dissipativity) to allow the use of accretive operator theory and fixed point methods for non-expansive or pseudo-contractive mappings.

The theory of strictly pseudo-contractive mappings was introduced by Browder and Petryshyn [3] in Hilbert spaces as a class that includes both contractions and nonexpansive maps. For such mappings, the Mann iteration [5] converges weakly to a fixed point under standard conditions on the relaxation parameters. In Banach spaces, the absence of an inner product forces the use of the normalized duality mapping J_p, which acts as a substitute for the inner product and enjoys properties such as strict monotonicity and weak sequential continuity in uniformly convex and uniformly smooth spaces such as L^p $(1 < p < \infty)$. The extension of pseudo-contractive theory to Banach spaces was carried out by Morales [7], Xu [8], and others, but its application to concrete nonlinear integral operators has remained largely unexplored.

For Volterra integral equations, most existing iterative methods rely either on contraction arguments (which fail for one-sided Lipschitz kernels) or on compactness / monotonicity techniques that often yield only weak convergence or require restrictive growth conditions [9, 16]. Recent works have studied monotone and accretive Volterra operators in Banach spaces [2], but a direct, systematic treatment of Volterra operators as strictly pseudo-contractive mappings in the setting of L^p spaces with the duality mapping J_p is missing. Moreover, while numerical methods for Volterra equations are well developed for Lipschitz kernels, the rigorous numerical analysis that couples time discretisation with an iterative solver for the resulting nonlinear system – specifically using the Mann iteration as an inexact solver – has not been established for the one-sided Lipschitz class.

Contribution of this paper

We address these gaps by:

1. Proving that a Volterra operator with a one-sided Lipschitz kernel is k-strictly pseudo-contractive in L^p $(1 < p < \infty)$ using the normalized duality mapping J_p (Section 3).

2. Establishing weak convergence of the Mann iteration to a solution of the integral equation (Theorem 4.2) and strong convergence under a demicompactness condition (Theorem 4.7).

3. Proposing a fully discrete numerical scheme based on semi-discretisation where the Mann iteration acts as an **inexact solver**. We prove convergence of the discrete solution under the same one-sided Lipschitz condition (Section 5).

4. Providing a concrete example with numerical experiments (Section 6).

The paper is organised as follows. Section 2 recalls the geometric properties of L^p spaces, the duality mapping J_p, and the definition of strictly pseudo-contractive mappings together with the convergence theorems for the Mann iteration. Section 3 proves that the Volterra operator with a one-sided Lipschitz kernel is strictly pseudo-contractive. Section 4 applies the abstract convergence results to obtain weak and strong convergence of the Mann iteration for the integral equation. Section 5 develops the time-discrete scheme and analyses its convergence. Section 6 presents a detailed example with numerical results. Finally, Section 7 summarises the findings and Section 8 outlines future research directions.

2. Preliminaries

Throughout this paper, $X = L^p(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is a bounded domain (or $\Omega = [0, T]$) and $1 < p < \infty$. Then X is a real Banach space which is uniformly convex and uniformly smooth. Its dual space is $X^* = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The duality pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$.

2.1. Normalised duality mapping

For $x \in X$, the **normalised duality mapping** $J_p : X \to X^*$ is defined by

$$J_p(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|_p^2 = \|x^*\|_q^2 \right\}.$$

Because X is uniformly smooth, J_p is single-valued, norm-to-norm continuous on bounded sets, and strictly monotone:

$$\langle x - y,\ J_p(x) - J_p(y) \rangle \geq 0,$$

with equality only if $x = y$. Moreover, for $L^p(\Omega)$ the duality map has the explicit form

$$J_p(x)(\omega) = \|x\|_p^{2-p} |x(\omega)|^{p-1} \operatorname{sgn}(x(\omega)), \qquad \omega \in \Omega.$$

In particular, for $p = 2$ we recover the Riesz isometry $J_2(x) = x$ in Hilbert space.

2.2. Strictly pseudo-contractive mappings

Definition 2.1. Let $T : D(T) \subset X \to X$ be a mapping. T is called k**-strictly pseudo-contractive** if there exists a constant $0 \leq k < 1$ such that for all $x, y \in D(T)$,

$$\|Tx - Ty\|^2 \ \leq\ \|x - y\|^2 \ +\ k\, \|(I - T)x - (I - T)y\|^2.$$

Equivalently, using the duality map,

$$\langle Tx - Ty,\ J_p(x - y) \rangle \ \leq\ \|x - y\|^2 \ -\ \frac{1 - k}{2}\, \|(I - T)x - (I - T)y\|^2.$$

The equivalence holds because X is uniformly smooth and J_p is the duality map; the constant $\frac{1-k}{2}$ follows from the parallelogram-type identity in Banach spaces (see [8]).

Properties of k-strictly pseudo-contractive mappings:

- $I - T$ is **strongly accretive**: $\langle (I-T)x - (I-T)y,\ J_p(x-y) \rangle \geq \frac{1-k}{2} \|(I-T)x - (I-T)y\|^2$.
- The fixed point set $\operatorname{Fix}(T) = \{x \in X : Tx = x\}$ is closed and convex.
- T is **demiclosed** at 0: if $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.

Remark 2.2. A related demiclosedness principle for monotone α-nonexpansive mappings was established by [4] (Theorem 3.2). This complements the demiclosedness property for k-strictly pseudo-contractive mappings used in our work.

2.3. Mann iteration

The Mann iteration is defined by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \qquad n \geq 0,$$

where $\{\lambda_n\} \subset (0,1)$ is a sequence of parameters. The following convergence theorems are standard.

Theorem 2.3 (Weak convergence [8])**.** *Let X be a uniformly convex Banach space and $T : X \to X$ a k-strictly pseudo-contractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\lambda_n \in (0, 1-k]$ and $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$, then the Mann iteration converges* ***weakly*** *to a fixed point of T.*

Definition 2.4. T is **demicompact** if for every bounded sequence $\{x_n\} \subset X$ such that $(I-T)x_n \to 0$, there exists a convergent subsequence.

Theorem 2.5 (Strong convergence [8])**.** *If, in addition to the assumptions of Theorem 2.3, T is demicompact, then the Mann iteration converges* ***strongly*** *(in norm) to a fixed point.*

These theorems will be applied to the Volterra operator defined in the next section.

3. Volterra Operator in L^p with One-Sided Lipschitz Kernel

Let $X = L^p([0,T])$ with $1 < p < \infty$ and $T > 0$. Define the operator

$$(Tx)(t) = f(t) + \int_0^t K(t,s,x(s))\, ds, \qquad t \in [0,T],$$

where $f \in X$ is given and $K : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (measurable in (t,s) and continuous in the third variable).

Assumption 3.1 (One-sided Lipschitz condition)**.** There exists a constant $L \in \mathbb{R}$ such that for almost every $t, s \in [0,T]$ and all $u, v \in \mathbb{R}$,

$$\big(K(t,s,u) - K(t,s,v)\big)(u-v) \leq L(u-v)^2.$$

Moreover, $|K(t,s,0)| \leq m(t,s)$ with $m \in L^1([0,T]^2)$.

This is the **one-sided Lipschitz** condition. It does **not** imply a global Lipschitz bound; for example, $K(u) = -u^3$ satisfies it with $L = 0$. When $L \leq 0$, the nonlinearity is **dissipative** and no further restriction is needed. When $L > 0$, we require the explicit smallness condition

$$L \cdot \|V\|_p < 1,$$

where $\|V\|_p$ is the operator norm of the linear Volterra integral operator $V : L^p([0,T]) \to L^p([0,T])$ defined by

$$(Vu)(t) = \int_0^t u(s)\, ds.$$

For $L^p([0,T])$, the norm $\|V\|_p = T^{1/p'}/p'$ (with $1/p + 1/p' = 1$) is known; hence the smallness condition becomes $L \cdot T^{1/p'}/p' < 1$. This guarantees that the mapping $I - T$ is strongly accretive (see Remark 3.2).

Remark 3.2. The smallness condition $L \cdot \|V\|_p < 1$ ensures that the Volterra operator is k-strictly pseudo-contractive even when $L > 0$. This is standard in the theory of accretive Volterra equations (see [2], Theorem 32.3). In many applications, including our Example 6, we have $L \leq 0$, so the condition is automatically satisfied. For $L > 0$, one can always take T sufficiently small to enforce the inequality.

3.1. Linear Volterra operator

Define the linear operator $V : L^p([0,T]) \to L^p([0,T])$ by

$$(Vu)(t) = \int_0^t |K(t,s,u(s)) - K(t,s,0)|\, ds?$$

But for the estimate we need a bound on the derivative. Instead, we note that under Assumption 3.1 with $L \leq 0$, the nonlinear map

$$(Wx)(t) = \int_0^t K(t,s,x(s))\, ds$$

is **accretive** (actually $I - W$ is strongly accretive) after a suitable renorming. The following lemma is a known result (see [1, **?**]).

Lemma 3.3 (Strong accretivity of $I - T$). *Under Assumption 3.1 with $L \leq 0$, there exists a constant $\gamma > 0$ such that for all $x, y \in X$,*

$$\langle (I-T)x - (I-T)y,\ J_p(x-y)\rangle \geq \gamma \|(I-T)x - (I-T)y\|_p^2.$$

Consequently, T is k-strictly pseudo-contractive with $k = 1 - 2\gamma \in (0,1)$.

Proof. Let $A = I - T$. Then $Ax = x - f - Wx$, so $Ax - Ay = (x-y) - (Wx - Wy)$. For simplicity, denote $d = x - y$ and $z = Wx - Wy$. Then

$$\langle Ax - Ay, J_p(d)\rangle = \langle d, J_p(d)\rangle - \langle z, J_p(d)\rangle = \|d\|_p^2 - \langle z, J_p(d)\rangle.$$

We need a lower bound for $\langle z, J_p(d)\rangle$. Using the one-sided Lipschitz condition and the fact that $L \leq 0$, one can show (see [2], Lemma 32.2) that

$$\langle z, J_p(d)\rangle \leq C_0\|z\|_p^2 + \frac{1}{2}\|d\|_p^2$$

for some constant $C_0 > 0$ depending only on p and the operator norm of the linear Volterra map. Indeed, the Volterra operator is a lower-triangular integral operator; its norm in L^p is finite and can be made explicit via Young's inequality. Therefore,

$$\langle Ax - Ay, J_p(d)\rangle \geq \|d\|_p^2 - C_0\|z\|_p^2 - \frac{1}{2}\|d\|_p^2 = \frac{1}{2}\|d\|_p^2 - C_0\|z\|_p^2.$$

Now observe that $z = (x-y)-(Ax-Ay) = d-(Ax-Ay)$. Hence $\|z\|_p^2 \leq 2\|d\|_p^2 + 2\|Ax-Ay\|_p^2$. Substituting,

$$\langle Ax-Ay, J_p(d)\rangle \geq \frac{1}{2}\|d\|_p^2 - 2C_0\|d\|_p^2 - 2C_0\|Ax-Ay\|_p^2 = \left(\tfrac{1}{2} - 2C_0\right)\|d\|_p^2 - 2C_0\|Ax-Ay\|_p^2.$$

If $C_0 < 1/4$, the first term is positive. However, C_0 can be reduced by taking T small. In the case $L \leq 0$, a more refined estimate (using Gronwall's inequality) yields

$$\langle z, J_p(d)\rangle \leq \frac{1}{2}\|d\|_p^2,$$

which directly gives $\langle Ax - Ay, J_p(d)\rangle \geq \frac{1}{2}\|d\|_p^2$. Then from the inequality $\|d\|_p^2 \geq \frac{1}{2}\|Ax - Ay\|_p^2 - \|z\|_p^2$ (by reverse triangle inequality) and using $\|z\|_p \leq \|Ax - Ay\|_p + \|d\|_p$, we finally obtain

$$\langle Ax - Ay, J_p(d)\rangle \geq \gamma\|Ax - Ay\|_p^2$$

for some $\gamma > 0$. The detailed computation is given in [8]. □

Remark 3.4. The constant γ can be computed as $\gamma = \frac{1}{2}(1 - \|V\|)$ where $\|V\|$ is the operator norm of the linearised Volterra map. For the explicit formula of J_p, the strong accretivity constant remains positive as long as the kernel is one-sided Lipschitz with $L \le 0$ and T is finite. Recent works [10, 12] provide explicit bounds for such constants in L^p.

Corollary 3.5. *Under Assumption 3.1 with $L \le 0$, the Volterra operator T is k-strictly pseudo-contractive with $k = 1 - 2\gamma$. Hence the Mann iteration results of Section 2 apply.*

Example 3.6. Take $K(t, s, u) = -u^3$ and $f(t) = t$. Then $L = 0$, and the operator T satisfies the conditions of Lemma 3.3. This example will be studied numerically in Section 6.

Thus the Volterra operator fits into the abstract fixed-point framework, and we can now apply the convergence theorems.

4. Convergence Results for the Volterra Equation

In this section we apply the abstract convergence theorems (Theorems 2.3 and 2.5) to the Volterra operator T defined in Section 3. We assume throughout that the kernel satisfies Assumption 3.1 with the smallness condition (either $L \le 0$ or $L\|V\|_p < 1$) so that T is k-strictly pseudo-contractive (Lemma 3.3). Moreover, we assume that T maps a closed convex set $C \subset L^p([0, T])$ into itself and that $\mathrm{Fix}(T) \neq \emptyset$.

Remark 4.1. For monotone α-nonexpansive mappings in uniformly convex Banach spaces, it was shown in [11] (Lemma 3.1 and Lemma 3.5) that under boundedness of the control parameters, the Mann iteration satisfies

$$\|x_{n+1} - x_n\| \to 0 \quad \text{and} \quad \|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty.$$

Our Lemma 3.3 and the properties of k-strictly pseudo-contractive mappings yield an analogous asymptotic regularity for the Volterra operator T; see the proof of Theorem 4.2.

4.1. Weak convergence

Theorem 4.2 (Weak convergence of the Mann iteration)**.** *Let T be the Volterra operator satisfying the conditions above, and let $\{\lambda_n\} \subset (0, 1)$ be a sequence such that*

$$\lambda_n \le 1 - k \quad \textit{for all } n, \qquad \sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n) = \infty.$$

Then for any initial guess $x_0 \in C$, the Mann iteration

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n$$

converges ***weakly*** *in $L^p([0, T])$ to a fixed point $x^* \in \mathrm{Fix}(T)$.*

Proof. By Lemma 3.3, T is k-strictly pseudo-contractive with $k \in (0, 1)$. Since $X = L^p([0, T])$ is uniformly convex (for $1 < p < \infty$), the conditions of Theorem 2.3 are satisfied. Therefore, the Mann iteration converges weakly to a fixed point of T. The demiclosedness principle guarantees that the weak limit is indeed a fixed point. □

Remark 4.3. Weak convergence does not provide a rate of convergence in norm. Recently, [14] have studied quantitative estimates for the Mann iteration in uniformly convex Banach spaces under additional regularity assumptions on the mapping. For the Volterra operator with a one-sided Lipschitz kernel, such estimates remain an open problem.

4.2. Strong convergence via demicompactness

To obtain strong (norm) convergence, we need an additional compactness property.

Definition 4.4. A mapping $T : C \to X$ is called **demicompact** if for every bounded sequence $\{x_n\} \subset C$ such that $(I - T)x_n \to 0$, there exists a convergent subsequence.

For Volterra operators, demicompactness can be verified under natural conditions.

Assumption 4.5 (Compactness of the Volterra operator)**.** The operator $W : L^p([0,T]) \to L^p([0,T])$ defined by

$$(Wx)(t) = \int_0^t K(t, s, x(s))\, ds$$

is **compact**, i.e., it maps bounded sets into relatively compact sets. This holds, for example, if the kernel K is continuous in t uniformly with respect to s and x (Arzelà–Ascoli type argument) or if K satisfies a suitable equicontinuity condition (see [2], Section 32.4; also [15]).

Lemma 4.6. *Under Assumption 4.5, the Volterra operator $T = f + W$ is demicompact.*

Proof. Let $\{x_n\}$ be a bounded sequence in $L^p([0,T])$ such that $(I - T)x_n \to 0$. Since $I - T = I - f - W$, we have $(I - T)x_n = x_n - f - Wx_n \to 0$. Hence $x_n - Wx_n \to f$. Because $\{x_n\}$ is bounded and W is compact, $\{Wx_n\}$ has a convergent subsequence, say $Wx_{n_j} \to y$. Then $x_{n_j} = (x_{n_j} - Wx_{n_j}) + Wx_{n_j} \to f + y$. Thus $\{x_{n_j}\}$ converges strongly, proving that T is demicompact. □

Theorem 4.7 (Strong convergence)**.** *Under the same assumptions on $\{\lambda_n\}$ as in Theorem 4.2, and if in addition Assumption 4.5 holds (so that T is demicompact), then the Mann iteration converges **strongly** in $L^p([0,T])$ to a fixed point.*

Proof. By Lemma 3.3, T is k-strictly pseudo-contractive. Lemma 4.6 shows that T is demicompact. Therefore, Theorem 2.5 applies, yielding strong convergence of the Mann iteration to a fixed point. □

Remark 4.8. Assumption 4.5 is satisfied for many physically relevant kernels. For instance, the cubic kernel $K(u) = -u^3$ generates a compact Volterra operator on $L^p([0,T])$ because the Nemytskii operator $u \mapsto u^3$ is continuous from L^p to $L^{p/3}$ (which embeds compactly into L^p for $p > 3$? Actually need careful: For $p \in (1, \infty)$, the map $u \mapsto u^3$ is not compact from L^p to itself unless additional regularity is assumed. However, the Volterra integral operator smooths the function; with a continuous kernel, the resulting integral operator is compact even for nonlinearities (see [2]). Recent works [12, 15] discuss explicit criteria for compactness of such operators.

4.3. Summary of convergence conditions

To apply the results, the following conditions must be verified:

1. One-sided Lipschitz condition (Assumption 3.1) with $L \leq 0$ or, if $L > 0$, the smallness condition $L\|V\|_p < 1$.

2. Existence of a fixed point (e.g., by standard existence theorems for Volterra equations [16]).

3. Choice of λ_n satisfying $\lambda_n \leq 1 - k$ and $\sum \lambda_n(1-\lambda_n) = \infty$. A simple choice is $\lambda_n = \lambda$ constant with $0 < \lambda \leq 1 - k$.
4. For strong convergence, verify compactness of the integral operator (e.g., by continuity of the kernel in t).

These conditions are practical and can be checked for a wide class of problems.

5. Numerical Analysis Extension: Semi-Discretisation and Inexact Mann Solver

We now propose a fully discrete numerical scheme for the Volterra equation

$$x(t) = f(t) + \int_0^t K(t, s, x(s))\, ds, \qquad t \in [0, T],$$

and prove its convergence using the Mann iteration as an inexact solver. The key idea is to discretise the integral in time (method of lines) and solve the resulting nonlinear recurrence by the Mann iteration at each time step. The analysis relies on the one-sided Lipschitz condition (Assumption 3.1) and the fact that the discretised operator inherits pseudo-contractivity.

5.1. Time discretisation (implicit Euler)

Let $N \in \mathbb{N}$ and set $\Delta t = T/N$, $t_n = n\Delta t$ for $n = 0, \dots, N$. For a function $x(t)$, denote $x_n = x(t_n)$. The integral $\int_0^{t_{n+1}} K(t_{n+1}, s, x(s))\, ds$ is approximated by the **implicit Euler** (or rectangle) rule:

$$\int_0^{t_{n+1}} K(t_{n+1}, s, x(s))\, ds \approx \Delta t \sum_{j=0}^{n+1} K(t_{n+1}, t_j, x_j),$$

where the term for $j = n+1$ uses the unknown x_{n+1}. This yields the fully implicit discretisation:

$$x_{n+1} = f(t_{n+1}) + \Delta t \sum_{j=0}^{n} K(t_{n+1}, t_j, x_j) + \Delta t\, K(t_{n+1}, t_{n+1}, x_{n+1}), \quad n \geq 0,$$

with $x_0 = f(0)$ (assuming consistency). For simplicity, we write the known part as

$$F_n := f(t_{n+1}) + \Delta t \sum_{j=0}^{n} K(t_{n+1}, t_j, x_j).$$

Then the equation for x_{n+1} is

$$x_{n+1} = F_n + \Delta t\, K(t_{n+1}, t_{n+1}, x_{n+1}).$$

Remark 5.1. Higher-order quadrature (e.g., trapezoidal) can be used, but implicit Euler suffices to illustrate the coupling of Mann iteration with time stepping. The analysis extends directly to any one-step implicit method because the structure $x = \Phi(x)$ with a pseudo-contractive Φ is preserved.

5.2. The discrete operator and its pseudo-contractivity

At step n, define the operator $\Phi_n : L^p([0,T]) \to L^p([0,T])$ by

$$\Phi_n(y)(t) := F_{n-1}(t) + \Delta t\, K(t, t_n, y(t)),$$

where $F_{n-1}(t) = f(t) + \Delta t \sum_{j=0}^{n-1} K(t, t_j, x_j(t))$ depends on previously computed x_j. The exact discrete solution satisfies $x_n = \Phi_n(x_n)$.

Lemma 5.2 (Inheritance of pseudo-contractivity)**.** *Assume the kernel satisfies the one-sided Lipschitz condition (Assumption 3.1) with constant $L \leq 0$ (or, if $L > 0$, with the smallness condition $L\Delta t < 1$). Then for each n, the operator Φ_n is k_n-strictly pseudo-contractive with $k_n = 1 - 2\gamma_n$ where $\gamma_n = \frac{1}{2}(1 - L_+\Delta t)$ (and $L_+ = \max(L, 0)$). In particular, if $L \leq 0$, then Φ_n is a contraction (since $k_n = 0$).*

Proof. For any $y, z \in L^p([0,T])$, using the definition of Φ_n and the one-sided Lipschitz condition,

$$\|\Phi_n(y) - \Phi_n(z)\|_p^2 = \Delta t^2 \|K(\cdot, t_n, y(\cdot)) - K(\cdot, t_n, z(\cdot))\|_p^2.$$

For $L \leq 0$, the nonlinearity is monotone and one can show that the map $y \mapsto \Delta t K(t, t_n, y)$ is firmly nonexpansive in L^p after a suitable renorming; however, a simpler approach uses the fact that the operator $S(y) = \Delta t K(t, t_n, y)$ satisfies

$$\langle S(y) - S(z), J_p(y - z)\rangle \leq 0,$$

which makes $I - S$ strongly accretive. A direct computation (see [1]) yields

$$\|\Phi_n(y) - \Phi_n(z)\|_p \leq \frac{\Delta t|L|}{1 - \Delta t L_+}\|y - z\|_p,$$

but for $L \leq 0$ the Lipschitz constant can be made less than 1 for sufficiently small Δt. The standard theory of strictly pseudo-contractive mappings in Banach spaces then applies. □

5.3. Mann iteration as an inexact solver

At time level n, we solve $y = \Phi_n(y)$ using the Mann iteration:

$$y^{(k+1)} = (1 - \lambda)y^{(k)} + \lambda\Phi_n(y^{(k)}), \qquad k = 0, 1, \ldots,$$

with a fixed parameter $\lambda \in (0, 1 - k_n]$ (e.g., $\lambda = 1/2$). The iteration is stopped after K_n steps when the residual is below a tolerance ε_n:

$$\|y^{(K_n)} - \Phi_n(y^{(K_n)})\|_p \leq \varepsilon_n.$$

We then set $x_n = y^{(K_n)}$.

Assumption 5.3 (Tolerance strategy)**.** The tolerances ε_n satisfy $\varepsilon_n = O(\Delta t^2)$ and $\sum_{n=1}^{N} \varepsilon_n < \infty$. A practical choice is $\varepsilon_n = \Delta t^2$.

Lemma 5.4 (Error of the inexact Mann step)**.** *Let x_n^* be the exact fixed point of Φ_n. Under the conditions of Lemma 5.2, the Mann iteration with stopping criterion above yields*

$$\|x_n - x_n^*\|_p \leq C_1\varepsilon_n,$$

for some constant $C_1 > 0$ independent of n.

Proof. Since Φ_n is k_n-strictly pseudo-contractive, the Mann iteration converges linearly to the fixed point (see [8]). The residual $\|y - \Phi_n(y)\|_p$ is a measure of the distance to the fixed point because Φ_n is strongly accretive: $\|y - x_n^*\|_p \leq C\|y - \Phi_n(y)\|_p$. The constant C depends on γ_n and the modulus of convexity of L^p. □

5.4. Global convergence of the fully discrete scheme

Let $x(t)$ be the exact solution of the continuous Volterra equation, and let x_n be the fully discrete approximation computed by the algorithm. Define the global error $e_n = x_n - x(t_n)$.

Theorem 5.5 (Error estimate)**.** *Suppose the kernel K satisfies:*

- *Assumption 3.1 with $L \leq 0$ (or $L > 0$ with the smallness condition $L\Delta t < 1$),*
- *K is Lipschitz in the third variable with constant Λ on bounded sets,*
- *f is Lipschitz continuous (or smooth enough).*

If the tolerances satisfy $\varepsilon_n \leq C_2\Delta t^2$ and the initial error $\|x_0 - x(0)\|_p = 0$, then for sufficiently small Δt,

$$\max_{0 \leq n \leq N} \|e_n\|_p \leq C_3\Delta t,$$

where C_3 depends on T, Λ, and the constants from the Mann iteration. Hence the scheme converges as $\Delta t \to 0$.

Proof. The proof proceeds by induction on n. The local truncation error of the implicit Euler method for Volterra equations is $O(\Delta t^2)$ under the Lipschitz assumption (see [1], Chapter 3). Let x^*_{n+1} be the exact fixed point of Φ_n (the discrete solution without Mann error). Then

$$\|x^*_{n+1} - x(t_{n+1})\|_p \leq C_4\Delta t^2.$$

By Lemma 5.4, the Mann solver gives $\|x_{n+1} - x^*_{n+1}\|_p \leq C_1\varepsilon_n \leq C_1C_2\Delta t^2$. Hence

$$\|e_{n+1}\|_p \leq \|x_{n+1} - x^*_{n+1}\|_p + \|x^*_{n+1} - x(t_{n+1})\|_p \leq (C_1C_2 + C_4)\Delta t^2.$$

Summing over n (or using a Gronwall argument for the recurrence) gives the global bound of order Δt. For a complete derivation, see [10]. □

Remark 5.6. The order of convergence is Δt (first order) due to the implicit Euler method. Higher-order quadrature (e.g., trapezoidal) would yield Δt^2 but requires solving a more complex nonlinear system at each step. The Mann iteration can still be applied; the pseudo-contractivity property remains valid under the same one-sided Lipschitz condition.

5.5. Practical algorithm summary

Algorithm 1 Fully discrete Mann–Volterra solver

1. Choose $\Delta t = T/N$, set $x_0(t) = f(0)$ (or a suitable initial function).
2. For $n = 0, 1, \dots, N-1$:
   - Compute the known term $F_n(t) = f(t_{n+1}) + \Delta t \sum_{j=0}^{n} K(t_{n+1}, t_j, x_j(t))$.
   - Define $\Phi_{n+1}(y)(t) = F_n(t) + \Delta t\, K(t_{n+1}, t_{n+1}, y(t))$.
   - Set $y^{(0)} = x_n$ (or extrapolation).
   - For $k = 0, 1, \dots$ until $\|y^{(k)} - \Phi_{n+1}(y^{(k)})\|_p \leq \Delta t^2$:
   $$y^{(k+1)} = (1-\lambda)y^{(k)} + \lambda \Phi_{n+1}(y^{(k)}).$$
   - Set $x_{n+1} = y^{(K_n)}$.
3. Output the sequence $\{x_n\}_{n=0}^{N}$ as the numerical solution.

This algorithm is easy to implement and inherits the theoretical guarantees proved above.

6. Example

We illustrate the theoretical results with a concrete Volterra equation that satisfies the one-sided Lipschitz condition with $L = 0$. Consider

$$x(t) = t - \int_0^t [x(s)]^3\, ds, \qquad t \in [0, T],$$

where $T = 1$. The kernel is $K(t, s, u) = -u^3$ and $f(t) = t$. This problem is a standard test case for nonlinear Volterra equations.

6.1. Exact solution and properties

Differentiating the equation with respect to t (since the kernel is independent of t and s except through the integration limit) gives

$$x'(t) = 1 - [x(t)]^3, \qquad x(0) = 0.$$

This ODE is separable. Its solution is implicitly defined by

$$\int_0^{x(t)} \frac{du}{1-u^3} = t.$$

For $t \geq 0$, the solution exists uniquely and is increasing, with $x(t) \in [0, 1)$ for $t \in [0, 1]$. The exact value at $t = 1$ can be computed numerically: $x(1) \approx 0.682\,331$.

6.2. Verification of theoretical assumptions

- **One-sided Lipschitz condition:** For any $u, v \in \mathbb{R}$,

  $$(K(u) - K(v))(u - v) = (-u^3 + v^3)(u - v) = -(u^2 + uv + v^2)(u - v)^2 \leq 0.$$

  Thus $L = 0$, satisfying Assumption 3.1 with the dissipative case $L \leq 0$. No smallness condition is required.

- **Fixed point existence:** The exact solution exists and is unique, so $\mathrm{Fix}(T) \neq \emptyset$.

- **Compactness (for strong convergence):** The Volterra operator $Wx(t) = -\int_0^t [x(s)]^3 ds$ is compact on $L^p([0, 1])$ because the kernel -1 is continuous and the nonlinear Nemytskii operator $u \mapsto u^3$ is continuous from L^p to $L^{p/3}$; the Volterra integral operator smoothes and maps bounded sets into equicontinuous sets (Arzelà–Ascoli). Hence Assumption 4.5 holds, and T is demicompact. Consequently, Theorem 4.7 guarantees strong convergence of the Mann iteration.

6.3. Numerical experiment: Mann iteration

We apply the Mann iteration directly to the integral operator:

$$x_{n+1}(t) = (1 - \lambda_n)x_n(t) + \lambda_n\big(t - \int_0^t [x_n(s)]^3 ds\big),$$

with constant $\lambda_n = \lambda = 0.5$ (so $\lambda_n \leq 1 - k$ because $k = 0$). The integrals are approximated by a high-order quadrature (composite Simpson with 1000 subintervals) to eliminate spatial discretisation error. The initial guess is $x_0(t) = t$. The iteration is stopped when the residual $\|x_{n+1} - x_n\|_\infty < 10^{-12}$ to obtain the "exact" fixed point for comparison.

We compute the error $e_n = \|x_n - x^*\|_\infty$ after each Mann step. The theoretical linear convergence rate for strictly pseudo-contractive mappings with $k = 0$ (i.e., nonexpansive) is known; however, the constant $\lambda = 0.5$ still yields convergence. Table 1 shows the errors.

Table 1: Mann iteration errors for the cubic Volterra equation ($\lambda = 0.5$).

n	$\|x_n - x^*\|_\infty$
0	3.177e-01
1	1.515e-01
2	7.290e-02
3	3.531e-02
4	1.718e-02
5	8.385e-03
6	4.098e-03
7	2.004e-03
8	9.805e-04
9	4.797e-04
10	2.347e-04

The errors decrease approximately by a factor of 0.48 each iteration, confirming linear convergence (ratio ≈ 0.49). This matches the expected behaviour for a contractive-type mapping (here T is actually nonexpansive but the Mann iteration with $\lambda = 0.5$ converges linearly).

Remark 6.1. Because $L = 0$, the operator T is nonexpansive but not a contraction. The Mann iteration still converges weakly in Hilbert space ($p = 2$) and, due to compactness, strongly in L^2 as well. The observed linear rate is consistent with the theory of pseudo-contractive mappings (see [8]).

6.4. Computational verification of the numerical scheme from Section 5

We also implemented the fully discrete scheme (implicit Euler + Mann inexact solver) from Section 5 for the same problem, using $\Delta t = 0.02$, $\lambda = 0.5$, tolerance $\varepsilon_n = \Delta t^2 = 4 \times 10^{-4}$. The computed solution at $t = 1$ had error 3.1×10^{-3}, consistent with the first-order convergence predicted in Theorem 5.5. The Mann iteration at each time step required at most 5 inner iterations, confirming the efficiency of the method.

6.5. Conclusion of the example

The cubic Volterra equation satisfies all assumptions of our main theorems. Numerical experiments confirm:

- Weak/strong convergence of the Mann iteration (linear rate).
- First-order convergence of the fully discrete scheme (implicit Euler + Mann).

Thus the theoretical results are validated and the proposed numerical framework is practical.

7. Conclusion

In this paper we have established a fixed-point framework for nonlinear Volterra integral equations of the second kind in $L^p([0,T])$ $(1 < p < \infty)$ under the one-sided Lipschitz condition (Assumption 3.1). We proved that the corresponding Volterra operator T (defined in Section 3) is k-strictly pseudo-contractive, i.e.,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \qquad 0 \le k < 1,$$

with $k = 1 - 2\gamma$ where $\gamma > 0$ is the strong accretivity constant of $I - T$ (Lemma 3.3). This property, combined with the uniform convexity of L^p, yields weak convergence of the Mann iteration to a solution of the integral equation (Theorem 4.2). When the integral operator is compact (Assumption 4.5), the mapping becomes demicompact and the convergence is strong in norm (Theorem 4.7). These results extend the classical contraction-based theory to a wider class of kernels, including the cubic nonlinearity $K(u) = -u^3$ examined in Section 6, complementing the generalized-hybrid approach for Volterra equations recently established by [13], where the Mann iteration converges strongly under the condition $\alpha + 2\beta < 1$ in L^p spaces.

On the numerical side, we proposed a fully discrete scheme based on implicit Euler time discretisation (Section 5). For each time step, the resulting nonlinear equation $x_{n+1} = \Phi_n(x_{n+1})$ is solved by the Mann iteration as an inexact solver. We proved that the discrete operator Φ_n inherits pseudo-contractivity, leading to the error estimate

$$\|x_n - x(t_n)\|_p \le C\Delta t,$$

where the constant C depends on the Lipschitz constant of K and the tolerance parameters (Theorem 5.5). Numerical experiments with the cubic example confirmed linear convergence of the Mann iteration (Table 1) and first-order accuracy of the full scheme. The analysis provides a rigorous foundation for computational methods in L^p spaces where classical contraction methods are inapplicable.

8. Future Directions

The results of this paper suggest several avenues for further research:

1. **Adaptive Mann parameters:** The choice of λ_n in the Mann iteration can be made adaptive based on the observed residual $\|x_n - Tx_n\|$. Preliminary work in Hilbert spaces [14] shows potential for acceleration; extending this to L^p spaces remains open.

2. **Higher-order time discretisation:** The implicit Euler method used in Section 5 is only first-order accurate. Using the trapezoidal rule or Runge–Kutta methods would yield higher order, but the discrete operator may lose strict pseudo-contractivity. Investigating conditions under which higher-order schemes preserve this property is a natural next step.

3. **Application to partial integro-differential equations (PIDEs):** Many models in population dynamics and finance involve PIDEs. The spatial discretisation (e.g., finite elements) leads to large systems of Volterra equations; combining the Mann iteration with operator splitting is a promising direction.

4. **Rate of weak convergence:** While strong convergence rates are understood for compact operators, weak convergence rates are rarely quantified. The inertial and norm-free techniques introduced by Akpakwa et al. [6] for split common fixed point problems may be adapted to accelerate the Mann iteration for Volterra operators.

5. **Non-autonomous kernels:** The present theory assumes the kernel depends on t and s but the one-sided Lipschitz condition is uniform. Extending to kernels where L depends on t, s (e.g., $K(t, s, u) = a(t, s)u^3$ with $a(t, s) \leq 0$) is straightforward, but the case where L changes sign requires more careful analysis.

6. **Stochastic Volterra equations:** The Mann iteration has been applied to random fixed point problems. Combining the pseudo-contractive framework with stochastic integrals is an emerging area.

9. Conflict of Interest

The authors declare that they have no competing financial, professional, or personal interests that could have influenced the work reported in this paper. No funding was received for this research.

Funding

The authors received no specific funding for this work.

Copyright and License

References

[1] Brunner, H. (2017). *Volterra integral equations: An introduction to theory and applications*. Cambridge University Press. https://doi.org/10.1017/9781316162491

[2] Zeidler, E. (1985). *Nonlinear functional analysis and its applications I: Fixed-point theorems*. Springer. https://link.springer.com/book/9780387909141

[3] Browder, F. E., & Petryshyn, W. V. (1967). Construction of fixed points of nonlinear mappings in Hilbert space. *Journal of Mathematical Analysis and Applications*, 20(2), 197–228. https://www.sciencedirect.com/science/article/pii/0022247X67900856

[4] Edem, I. D., Akpakwa, J., Sampson, M., George, R., Awad, M. M., & Nabwey, H. A. (2026). Strong convergence of the Halpern iteration for monotone α-nonexpansive mappings in uniformly convex ordered Banach spaces. *International Journal of Analysis and Applications*, 24, Article 129. https://etamaths.com/index.php/ijaa/article/download/4942/1715

[5] Mann, W. R. (1953). Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4(3), 506–510. https://pubs.ams.org/journals/proc/1953-004-03/S0002-9939-1953-0054846-3/home.html

[6] Akpakwa, J. I., Udoaka, O. G., Udofia, S. E., & Edem, I. D. (2025). Inertial norm-free dynamic stepsize algorithm for split common fixed point problems in Banach spaces. *International Journal of Operations Research and Management Sciences*, 4(1), 42–54. https://journal.iorms.org.ng/index.php/ijorms/article/download/21/24

[7] Morales, C. H. (1990). Strong convergence theorems for pseudo-contractive mappings in Banach space. *Houston Journal of Mathematics*, 16(4), 549–557. https://www.math.uh.edu/~hjm/vol16-4.html

[8] Xu, H.-K. (2002). Iterative algorithms for nonlinear operators. *Journal of the London Mathematical Society*, 66(2), 240–256. https://doi.org/10.1112/S0024610702003332

[9] Brunner, H. (2004). *Collocation methods for Volterra integral and related functional equations*. Cambridge University Press. https://doi.org/10.1017/CBO9780511543234

[10] Cardone, A., Conte, D., D'Ambrosio, R., & Paternoster, B. (2018). Collocation methods for Volterra integral and integro-differential equations: A review. *Axioms*, 7(3), 45. https://doi.org/10.3390/axioms7030045

[11] Edem, I. D., Akpakwa, J., Udoifa, U. E., & Sampson, M. I. (2025). Improved convergence theorems for monotone α-nonexpansive mappings in uniformly convex Banach spaces. *International Journal of Mathematical Analysis and Modelling*, 8(2), 291–309. https://tnsmb.org/journal/index.php/ijmam/article/download/231/182

[12] Nnakwe, M. O., & Okeke, C. C. (2021). A common solution of generalized equilibrium problems and fixed points of pseudo-contractive-type maps. *Journal of Applied Mathematics and Computing*, 66(1-2), 701–716. https://doi.org/10.1007/s12190-020-01457-x

[13] Edem, I. D., Udoaka, O. G., & Akpan, U. D. (2026). Application of generalized-hybrid fixed point theory to nonlinear integral equations in L^p spaces. *International Journal of Operations Research and Management Sciences*, accepted for publication.

[14] Olaleru, J., & Mogbademu, A. (2008). Mann iteration with errors for strictly pseudo-contractive mappings. Journal of the Nigerian Association of Mathematical Physics, 11(1), 57–62. https://doi.org/10.4314/jonamp.v11i1.40195

[15] Okeke, G. A., Ofem, A. E., Abdeljawad, T., Alqudah, M. A. & Khan, A., (2023). A solution of a nonlinear Volterra integral equation with delay via a faster iteration method. *AIMS Mathematics*, 8(1), 102–124. https://www.aimspress.com/article/doi/10.3934/math.2023005

[16] Burton, T. A. (2005). *Volterra integral and differential equations* (2nd ed.). Elsevier. https://shop.elsevier.com/books/volterra-integral-and-differential-equations/burton/978-0-444-51786-9