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Joukowski Mapping of Circular Domains to Aerodynamic Profiles

Etim Uduak James

Akwa Ibom State University, Nigeria

Corresponding Author: uduakjamesetim@gmail.com

Abstract. *Conformal mapping is one of the central geometric techniques of complex analysis because it permits a complicated planar domain to be represented by a simpler one while preserving local angles. This paper develops the analytic conditions for conformality from the principal linear part of a differentiable complex mapping, relates the Cauchy–Riemann equations to local rotation and dilation, and presents explicit mappings between standard domains. Particular attention is given to the Joukowski transformation and its constructive action on circular boundaries. Actual mappings are worked out for the upper half-plane and unit disk, for a strip and half-plane, and for circles mapped by the Joukowski transformation into a line segment, an ellipse, and an airfoil-like profile. The derivations are accompanied by graphical realizations and a discussion of potential-flow aerodynamics. The results make explicit the connection between the analytic formula, its critical points, transformed geometry, and the distinction between local conformality and global one-to-one behavior.*

Keywords: conformal mapping; Joukowski transformation; circular domains; airfoil profile; complex analysis; potential flow; aerodynamics.

1. Introduction

A conformal map is, informally, a transformation that preserves angles and their orientation at sufficiently small scales. In complex-variable form one writes

$$w = f(z) = u(x, y) + iv(x, y), \qquad z = x + iy, \tag{1}$$

where the real-valued functions u and v describe the corresponding planar transformation. The basic local picture is that a differentiable mapping can be replaced near a point by its principal linear part. When that linear part is a rotation followed by a positive scaling, the mapping is conformal. This local geometric viewpoint is standard in complex analysis [1, 5, 14, 16] and is also the central framework in the supplied source material, which treats conformality through orthogonal linear transformations and nonvanishing derivatives.

Conformal maps are valuable because they transfer problems from geometrically difficult domains to domains on which analytic or computational treatment is easier. Classical applications include electrostatics, heat conduction, ideal fluid flow, free-boundary problems and two-dimensional aerodynamics [2, 10, 13, 15]. In potential flow, the Joukowski transformation gives a particularly transparent example: a flow around a circle in an auxiliary complex plane can be transformed into a flow around a plate, ellipse or airfoil-like boundary.

The use of transformations to reveal structure is a recurring idea across mathematics. Although algebraic group structure and character-theoretic questions are different from conformal mapping, recent work by John, Udoaka and Musa [11] and by John, Udoaka and Udoakpan [12] illustrates the broader mathematical value of structural transformations, invariants and algebraic organization. These works are cited here as part of that broader mathematical context rather than as sources for the theory of conformal mapping.

The aim of this paper is to give a compact but publication-oriented treatment of the theory and, importantly, to include explicit domain-to-domain mappings and actual mapped curves rather than only descriptive statements.

2. Local Form of a Complex Mapping

Let $z_0 = x_0 + iy_0$ and $w_0 = f(z_0) = u_0 + iv_0$. If u and v are differentiable, then near (x_0, y_0),

$$u - u_0 \approx u_x(x - x_0) + u_y(y - y_0), \tag{2}$$

$$v - v_0 \approx v_x(x - x_0) + v_y(y - y_0). \tag{3}$$

Thus the principal linear part is represented by the Jacobian matrix

$$J_f(z_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{z=z_0}. \tag{4}$$

For a general real linear transformation, circles may become ellipses. For a conformal transformation, infinitesimal circles remain infinitesimal circles up to scale, and the angle between intersecting smooth curves is preserved [1, 6, 9].

If f is complex differentiable, the Cauchy–Riemann equations hold:

$$u_x = v_y, \qquad u_y = -v_x. \tag{5}$$

Consequently,

$$J_f = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = |f'(z)| \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{6}$$

where

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}, \qquad \alpha = \arg f'(z). \tag{7}$$

Hence $|f'(z)|$ is the local scale factor and $\arg f'(z)$ is the local angle of rotation [5, 14].

Theorem 2.1 (Local conformality)**.** *If f is analytic in a neighborhood of z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.*

Proof. Taylor expansion gives

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(|z - z_0|). \tag{8}$$

Multiplication by the nonzero complex number $f'(z_0) = \rho e^{i\alpha}$ scales lengths by ρ and rotates directions by α. The higher-order term vanishes relative to $|z - z_0|$, so angles are preserved locally. □

A critical point $f'(z_0) = 0$ requires separate treatment. For example, $f(z) = z^n$ has $f'(0) = 0$ for $n \geq 2$ and multiplies arguments by n near the origin; it is therefore not conformal at 0. Local conformality also does not automatically imply global injectivity. For instance, e^z has nonzero derivative everywhere but is periodic and thus not one-to-one on the full plane [6, 8].

3. Elementary Conformal Mappings

Several mappings serve as building blocks for more complicated constructions. Table 1 summarizes some of the most useful examples.

Table 1: Some elementary conformal mappings.

Mapping	Formula	Principal geometric effect
Affine	$w = az + b$, $a \neq 0$	rotation, dilation and translation
Power	$w = z^n$	multiplies arguments by n; conformal away from $z = 0$
Exponential	$w = e^z$	maps horizontal strips to angular sectors or slit planes
Inversion	$w = 1/z$	maps generalized circles to generalized circles
Möbius	$w = (az + b)/(cz + d)$	maps lines/circles to lines/circles
Joukowski	$w = z + b^2/z$	maps circles to plates, ellipses and airfoil-like profiles

3.1. Actual mapping: upper half-plane to unit disk

Consider

$$\boxed{w = \frac{z - i}{z + i}}. \tag{9}$$

Let $z = x + iy$ with $y > 0$. Then

$$|w|^2 = \frac{|z - i|^2}{|z + i|^2} = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2} < 1. \tag{10}$$

Therefore the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ maps into the unit disk $\mathbb{D} = \{w : |w| < 1\}$. If $y = 0$, then $|w| = 1$, so the real axis maps to the unit circle. Specific points are

$$z = i \mapsto 0, \qquad z = 0 \mapsto -1, \qquad z \to \infty \mapsto 1. \tag{11}$$

The inverse map is

$$z = i\frac{1 + w}{1 - w}. \tag{12}$$

This is a standard bilinear mapping and provides an explicit realization of boundary correspondence [1, 15, 16].

3.2. Actual mapping: a horizontal strip to the upper half-plane

Let

$$S = \{z = x + iy : 0 < y < a\} \tag{13}$$

and define

$$\boxed{w = e^{\pi z/a}}. \tag{14}$$

Since

$$w = e^{\pi x/a} \left(\cos \frac{\pi y}{a} + i \sin \frac{\pi y}{a} \right), \tag{15}$$

we have $0 < \arg w < \pi$ when $0 < y < a$. Hence S maps conformally onto the upper half-plane. The boundary $y = 0$ maps to the positive real axis, while $y = a$ maps to the negative real axis. This illustrates how the exponential function "opens" a strip into a half-plane [9, 14].

4. Bilinear Transformations

A bilinear or Möbius transformation is

$$w = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0. \tag{16}$$

Its derivative is

$$f'(z) = \frac{ad - bc}{(cz + d)^2}, \tag{17}$$

which is nonzero wherever the map is finite. Therefore it is conformal away from its pole. Möbius maps take generalized circles (circles or lines in the extended plane) to generalized circles and are completely determined by the images of three distinct points [1, 4, 5].

A convenient three-point construction is

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \tag{18}$$

This relation allows a desired map to be built directly from boundary correspondences.

5. The Joukowski Transformation

The Joukowski transformation is

$$\boxed{w = z + \frac{b^2}{z}}, \qquad b > 0, \ z \neq 0. \tag{19}$$

Its derivative is

$$\frac{dw}{dz} = 1 - \frac{b^2}{z^2} = \frac{z^2 - b^2}{z^2}. \tag{20}$$

The critical points are $z = \pm b$. Away from $z = 0, \pm b$, the transformation is locally conformal. The source material explicitly uses this mapping in aerodynamics to relate cylinder flows to flows around transformed bodies.

5.1. Circle mapped to a flat plate

Take the circle $|z| = b$ and write $z = be^{i\theta}$. Then

$$w = be^{i\theta} + \frac{b^2}{be^{i\theta}} = b(e^{i\theta} + e^{-i\theta}) \tag{21}$$
$$= 2b\cos\theta. \tag{22}$$

Thus the whole circle maps onto the real line segment

$$\boxed{-2b \le w \le 2b}. \tag{23}$$

This is an actual domain-boundary transformation, not merely a schematic interpretation.

Figure 1: The Joukowski map sends the unit circle to the line segment $[-2, 2]$.

5.2. A larger circle mapped to an ellipse

Let $z = ae^{i\theta}$ with $a > b$. Substitution into (19) gives

$$w = \left(a + \frac{b^2}{a}\right)\cos\theta + i\left(a - \frac{b^2}{a}\right)\sin\theta. \tag{24}$$

Writing $w = \xi + i\eta$,

$$\frac{\xi^2}{(a + b^2/a)^2} + \frac{\eta^2}{(a - b^2/a)^2} = 1. \tag{25}$$

Hence concentric circles with radius greater than b map to ellipses [2, 13].

5.3. Offset circle mapped to an airfoil-like profile

Now let

$$z(\theta) = c + Re^{i\theta}, \qquad c = c_x + ic_y, \tag{26}$$

where the circle is displaced from the origin. Applying (19) pointwise gives

$$w(\theta) = c + Re^{i\theta} + \frac{b^2}{c + Re^{i\theta}}. \tag{27}$$

For a suitable offset, the image becomes an airfoil-like closed curve. Figure 2 uses $b = 1$, $c = -0.10 + 0.08i$, and $R = 1.05$.

Figure 2: An actual numerical Joukowski mapping of an offset circle to an airfoil-like profile.

6. Potential Flow and Conformal Mapping

For a two-dimensional incompressible and irrotational flow, the complex potential may be written

$$F(z) = \phi(x, y) + i\psi(x, y), \tag{28}$$

where ϕ is the velocity potential and ψ is the stream function. The complex velocity is conventionally represented, depending on sign convention, by a derivative of F. Uniform flow, sources, sinks, vortices and doublets can all be expressed in complex form [2, 3, 13].

For example, a common cylinder-flow potential with circulation is

$$F(z) = U_\infty \left(z + \frac{R^2}{z} \right) + \frac{i\Gamma}{2\pi} \log z, \tag{29}$$

up to the adopted sign convention for circulation. If the geometry is transformed by $w = w(z)$, then the velocity transforms by the chain rule:

$$\frac{dF}{dw} = \frac{dF/dz}{dw/dz}. \tag{30}$$

For the Joukowski map,

$$\frac{dw}{dz} = 1 - \frac{b^2}{z^2}. \tag{31}$$

The vanishing of this derivative at $z = \pm b$ explains why the treatment of the trailing edge requires care. In classical airfoil theory, the Kutta condition selects the circulation so that the flow leaves the trailing edge smoothly; this leads to the Kutta–Joukowski lift relation [2, 3, 13]

$$L' = \rho U_\infty \Gamma, \tag{32}$$

where L' is lift per unit span, ρ is fluid density and Γ is circulation.

7. Existence, Uniqueness and Global Issues

The Riemann mapping theorem states that every nonempty simply connected proper open subset of $\mathbb{C}$ is conformally equivalent to the unit disk. With a suitable normalization, the conformal map is unique [1, 6, 8]. The theorem guarantees existence but generally does not provide a simple closed-form formula, which is why elementary mappings, Schwarz–Christoffel techniques, numerical conformal mapping and composition of known maps remain important [7, 10].

Three distinctions are important:

(i) $f'(z_0) \neq 0$ gives local conformality at z_0;

(ii) analyticity plus nonvanishing derivative on a domain does not by itself guarantee global injectivity;

(iii) topology constrains possible global conformal equivalences, so a multiply connected domain cannot in general be mapped one-to-one onto a simply connected domain.

These points prevent the common error of treating every analytic function as a globally conformal bijection.

8. Discussion

The calculations above show concretely how the geometry of a domain is encoded in an analytic function. The affine map reveals the local model: rotation, scaling and translation. Möbius transformations provide exact correspondences between disks, half-planes and generalized circles. Exponential maps open strips into half-planes, while the Joukowski map converts simple circular boundaries into physically meaningful aerodynamic shapes.

The explicit Joukowski computations are especially instructive. A circle at the critical radius maps to a degenerate ellipse, namely a line segment; a larger concentric circle maps to an ordinary ellipse; and an offset circle yields a cambered airfoil-like curve. This sequence makes visible the relation between analytic formula, derivative, critical points and transformed geometry.

From an applications standpoint, conformal mapping does not preserve all metric quantities: lengths and areas are generally changed. What is preserved locally is angle, with lengths scaled by $|f'(z)|$. Consequently, the method is particularly effective for equations and physical theories whose structure is compatible with conformal change, such as two-dimensional harmonic potential problems [10, 15].

9. Conclusion

This paper developed explicit Joukowski mappings of circular domains while placing the construction within the local theory of conformal maps, the Cauchy–Riemann equations and the geometric interpretation of the complex derivative. It then moved from theory to explicit constructions. The map $(z - i)/(z + i)$ was shown to take the upper half-plane to the unit disk; the exponential map was shown to take a horizontal strip to the upper half-plane; and the Joukowski transformation was calculated for a circle, an ellipse-generating circle and an offset circle producing an airfoil-like profile. These examples demonstrate that conformal mapping is both a rigorous theory of analytic functions and a constructive geometric method with direct relevance to potential flow and aerodynamics.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1979.

[2] J. D. Anderson, *Fundamentals of Aerodynamics*, 6th ed., McGraw–Hill Education, New York, 2017.

[3] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967.

[4] A. F. Beardon, *The Geometry of Discrete Groups*, Springer, New York, 1983.

[5] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th ed., McGraw–Hill Education, New York, 2014.

[6] J. B. Conway, *Functions of One Complex Variable I*, 2nd ed., Springer, New York, 1978.

[7] T. A. Driscoll and L. N. Trefethen, *Schwarz–Christoffel Mapping*, Cambridge University Press, Cambridge, 2002.

[8] P. L. Duren, *Univalent Functions*, Springer, New York, 1983.

[9] R. E. Greene and S. G. Krantz, *Function Theory of One Complex Variable*, 3rd ed., American Mathematical Society, Providence, 2006.

[10] P. Henrici, *Applied and Computational Complex Analysis, Vol. 1*, Wiley, New York, 1974.

[11] M. N. John, O. G. Udoaka and A. Musa, "Solvable Groups with Monomial Characters of Prime Power Codegree and Monolithic Characters," *Bulletin of Mathematics and Statistics Research*, vol. 11, 2023, pp. 98–102. doi:10.33329/bomsr.11.4.98.

[12] M. N. John, O. G. Udoaka and I. U. Udoakpan, "Group Theory in Lattice-Based Cryptography," *International Journal of Mathematics and Its Applications*, vol. 11, no. 4, 2023, pp. 111–125.

[13] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, 5th ed., Macmillan, London, 1968.

[14] T. Needham, *Visual Complex Analysis*, Oxford University Press, Oxford, 1997.

[15] Z. Nehari, *Conformal Mapping*, Dover Publications, New York, 1975.

[16] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, 3rd ed., Pearson, Upper Saddle River, 2003.

[17] E. M. Stein and R. Shakarchi, *Complex Analysis*, Princeton University Press, Princeton, 2003.

[18] L. N. Trefethen, *Approximation Theory and Approximation Practice*, extended ed., SIAM, Philadelphia, 2019.

[19] G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable: Theory and Technique*, SIAM, Philadelphia, 1983.

[20] S. D. Fisher, *Complex Variables*, 2nd ed., Dover Publications, Mineola, 1999.

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