DOI: 10.5281/zenodo.20749691 **Volume 1 | Issue 1 | ISSN: 3141-643X | Published: June 2026**

Ktrend - African Journal of Mathematics, Statistics and Computer Science

A Precision–Dispersion and Optimal Intervention Framework for Trachoma Transmission Dynamics under Random Environmental Fluctuations

OGOEGBULEM Ozioma[1,*], Angela Okwuolise OKONYE[1], Suit Patrick OGHENERHORO[2]

[1]Department of Mathematical Sciences, Dennis Osadebay University, Asaba, Delta State, Nigeria

[2]Department of Environmental Management and Toxicology, Dennis Osadebay University, Asaba, Delta State, Nigeria

Corresponding Author: Ozioma.Ogoegbulem@dou.edu.ng

Abstract

This study develops a new mathematical framework for analysing trachoma transmission dynamics through the integration of precision–dispersion measures, intervention effectiveness indices, and random environmental fluctuations. Unlike classical trachoma models that focus primarily on prevalence trajectories and reproduction numbers, the proposed framework quantifies the stability and reliability of intervention outcomes under uncertain epidemiological conditions. The human population is partitioned into susceptible, exposed, infectious, visually impaired, recovered, and protected classes, while environmental transmission effects are incorporated through a fluctuation-driven contact mechanism. A precision–dispersion functional is introduced to evaluate the consistency of intervention performance, and a stochastic fluctuation index is developed to measure variability arising from environmental and behavioural uncertainty. Fundamental model properties including positivity, boundedness, equilibrium existence, and stability conditions are established. The effective reproduction number is derived and sensitivity analysis is performed to identify dominant transmission pathways. Optimal control strategies involving hygiene compliance, antibiotic treatment, surgical intervention, and environmental sanitation are investigated using Pontryagin's Maximum Principle. Numerical simulations demonstrate that combined intervention strategies significantly reduce disease burden while maintaining greater stability under fluctuating epidemiological conditions. Cost-effectiveness analysis reveals that combined hygiene and treatment programmes provide the most favourable balance between epidemiological impact and implementation cost. The framework gives a new mathematical basis for evaluating infectious-disease interventions through simultaneous consideration of transmission dynamics, uncertainty, and intervention reliability.

Keywords: Trachoma; Precision–dispersion analysis; Optimal control; Random fluctuations; Cost-effectiveness analysis; Infectious disease modelling; Epidemiological stability.

1. Introduction

Trachoma remains one of the leading infectious causes of preventable blindness and continues to constitute a significant public-health burden in many developing regions. The disease is caused by repeated infection with *Chlamydia trachomatis*, which progressively damages the conjunctiva and may ultimately lead to irreversible visual impairment if untreated. Transmission persists in communities characterized by poor sanitation, inadequate hygiene, overcrowding, and limited access to healthcare services [21, 22, 23, 24].

Mathematical modelling has become an important tool for understanding infectious-disease transmission and evaluating intervention strategies. Classical compartmental models help estimate reproduction numbers, investigate equilibrium behaviour, and assess the effectiveness of control measures [5, 6, 9, 7]. Optimal-control methods further provide analytical mechanisms for designing intervention programmes that reduce disease prevalence subject to implementation cost [11, 12].

Most existing trachoma models focus primarily on deterministic transmission dynamics, reproduction number estimation, and optimal intervention policies [25, 26, 27]. While valuable, these approaches often assume that intervention outcomes are precisely measurable and unaffected by uncertainty. In practical settings, intervention effectiveness is influenced by environmental variation, behavioural variability, reporting errors, treatment compliance, demographic noise, and measurement uncertainties [13, 14].

Recent studies emphasize uncertainty quantification in mathematical biology and epidemiological modelling. Okeke et al. [1] modelled HIV infection of $CD4^+$ T cells using fractional-order derivatives and highlighted the usefulness of non-classical dynamics in biological systems with memory. Apanapudor et al. [3] estimated the impact of vaccination intervention on recovered coronavirus patients and demonstrated the significance of intervention-based modelling in public health. Ogoegbulem et al. [2] analysed data precision and dispersion under random noise fluctuations and established a framework for interpreting reliability in interacting systems. In a related statistical-learning study, Ogoegbulem, OGHENERHORO and OKONYE [4] developed precision, dispersion, and fluctuation measures for supervised learning algorithms, offering mathematical tools that motivate the present epidemiological extension.

Motivated by these developments, this study introduces a new mathematical framework for trachoma transmission analysis that incorporates precision–dispersion measures and stochastic fluctuation indices into a traditional epidemiological modelling structure. The aim is to quantify not only disease transmission and intervention effectiveness but also the reliability and stability of intervention outcomes under uncertain epidemiological conditions.

The main contributions of this work are as follows: (i) a precision–dispersion framework for trachoma transmission dynamics is developed; (ii) environmental uncertainty is incorporated through a random fluctuation mechanism; (iii) an intervention reliability functional is introduced; (iv) equilibrium and stability results are derived; (v) an optimal-control problem involving hygiene, treatment, and surgical intervention is formulated; and (vi) cost-effectiveness and sensitivity analyses are used to compare intervention strategies.

2. Literature Review

Mathematical models of infectious diseases have played a fundamental role in understanding transmission mechanisms, evaluating intervention policies, and supporting public-health planning [5, 16, 17, 29]. Classical SIR-type models provide threshold quantities such as the basic reproduction number, while extended compartmental models account for latency, recovery, protection, and disease complications [8, 7].

Trachoma transmission modelling has received increasing attention because of the disease's persistent burden in endemic regions. Epidemiological studies have shown that trachoma control depends on antibiotic treatment, facial cleanliness, environmental improvement, and surgery for advanced complications [22, 23, 24]. Age-structured and community-based models have also been proposed to evaluate mass drug administration and discontinuation thresholds [25, 26, 27].

Optimal-control theory has been widely applied in biological and epidemiological systems to identify policies that minimize infection burden while balancing the cost of intervention [11, 12]. In such models, Pontryagin's Maximum Principle is used to characterize time-dependent controls and adjoint variables. However, many control studies emphasize prevalence reduction and cost minimization without explicitly quantifying the stability of intervention outcomes under random perturbations.

Stochastic epidemic models address some of these limitations by incorporating randomness into transmission and transition processes [13, 14]. Nevertheless, the connection among stochastic fluctuations, intervention reliability, and dispersion of outcomes is still underdeveloped in trachoma modelling. The precision–dispersion concepts of Ogoegbulem et al. [2, 4] provide a useful bridge between statistical uncertainty and dynamic modelling. The present study extends this philosophy from simulated data and supervised learning systems to infectious-disease intervention dynamics.

3. Model Formulation and Precision–Dispersion Framework

Let

$$N(t) = S(t) + E(t) + I(t) + V(t) + R(t) + P(t)$$

denote the total population at time t, where $S(t)$ is the susceptible class, $E(t)$ is the exposed class, $I(t)$ is the infectious class, $V(t)$ is the visually impaired class, $R(t)$ is the recovered class, and $P(t)$ is the protected class.

To account for environmental uncertainty, the effective transmission rate is assumed to fluctuate according to

$$\beta(t) = \beta_0(1 + \eta(t)),$$

where $\beta_0 > 0$ is the baseline transmission coefficient and $\eta(t)$ is a random environmental fluctuation satisfying

$$E[\eta(t)] = 0, \qquad Var(\eta(t)) = \sigma^2.$$

The model is given by

$$\frac{dS}{dt} = \Lambda - \frac{\beta(t)SI}{N} - u_1 S - \mu S, \tag{1}$$

$$\frac{dE}{dt} = \frac{\beta(t)SI}{N} - (\kappa + \mu)E, \tag{2}$$

$$\frac{dI}{dt} = \kappa E - (\gamma + u_2 + \delta + \mu)I, \tag{3}$$

$$\frac{dV}{dt} = \delta I - (u_3 + \rho + \mu)V, \tag{4}$$

$$\frac{dR}{dt} = \gamma I + \rho V - \mu R, \tag{5}$$

$$\frac{dP}{dt} = u_1 S + u_2 I + u_3 V - \mu P. \tag{6}$$

Here $u_1(t)$ represents hygiene and sanitation intervention, $u_2(t)$ represents antibiotic treatment, and $u_3(t)$ represents surgical correction of severe complications.

3.1. Precision–Dispersion Measures

Let $X_1, X_2, \ldots, X_n$ denote percentage reductions in disease prevalence obtained from repeated intervention simulations. The mean effectiveness and variance are

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \sigma_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Following the precision–dispersion approach in [2, 4], define the dispersion index

$$D = \frac{\sigma_X^2}{\bar{X} + \varepsilon},$$

where $\varepsilon > 0$ prevents division by zero. The precision functional is

$$\mathcal{P} = \frac{\bar{X}}{\bar{X} + \sigma_X},$$

and the intervention reliability functional is

$$\mathcal{R}_I = \frac{\mathcal{P}}{1 + D}.$$

Small values of D indicate stable intervention performance, while large values indicate increased variability. Larger values of $\mathcal{R}_I$ correspond to more reliable intervention programmes.

4. Mathematical Analysis of the Model

4.1. Positivity of Solutions

Theorem 4.1. *If $S(0), E(0), I(0), V(0), R(0), P(0) \geq 0$, then all solutions of the model remain non-negative for all $t > 0$.*

Proof. From the susceptible equation,

$$\frac{dS}{dt} \geq -(u_1 + \mu)S.$$

Hence $S(t) \geq S(0)e^{-(u_1+\mu)t} \geq 0$. Similar differential inequalities apply to E, I, V, R, and P at their boundary values, so each compartment remains non-negative. Therefore the non-negative orthant is positively invariant. □

4.2. Boundedness

Adding the six state equations gives

$$\frac{dN}{dt} = \Lambda - \mu N.$$

Thus

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}),$$

and consequently $\limsup_{t\to\infty} N(t) \leq \Lambda/\mu$. The feasible region is therefore

$$\Omega = \left\{ (S, E, I, V, R, P) \in \mathbb{R}^6_+ : N \leq \frac{\Lambda}{\mu} \right\}.$$

4.3. Disease-Free Equilibrium and Effective Reproduction Number

At the disease-free equilibrium, $E = I = V = R = 0$, and

$$\mathcal{E}_0 = \left(\frac{\Lambda}{\mu + u_1}, 0, 0, 0, 0, \frac{u_1 \Lambda}{\mu(\mu + u_1)} \right).$$

Using the next-generation matrix method [8, 7], the infected compartments are E, I, V. The new-infection matrix and transition matrix are

$$F = \begin{pmatrix} 0 & \beta_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} \kappa + \mu & 0 & 0 \\ -\kappa & \gamma + u_2 + \delta + \mu & 0 \\ 0 & -\delta & u_3 + \rho + \mu \end{pmatrix}.$$

Thus the effective reproduction number is

$$R_e = \rho(FW^{-1}) = \frac{\beta_0 \kappa}{(\kappa + \mu)(\gamma + u_2 + \delta + \mu)}.$$

Theorem 4.2. *The disease-free equilibrium $\mathcal{E}_0$ is locally asymptotically stable whenever $R_e < 1$ and unstable whenever $R_e > 1$.*

Proof. The result follows from the next-generation theorem, since all eigenvalues associated with non-infected compartments have negative real parts and the infection subsystem is stable precisely when $\rho(FW^{-1}) < 1$. □

4.4. Endemic Equilibrium

When $R_e > 1$, a positive endemic equilibrium $\mathcal{E}^* = (S^*, E^*, I^*, V^*, R^*, P^*)$ exists. Solving the equilibrium equations gives

$$S^* = \frac{(\kappa+\mu)(\gamma+u_2+\delta+\mu)}{\beta_0\kappa},$$

$$E^* = \frac{\beta_0 S^* I^*}{N(\kappa+\mu)}, \qquad I^* = \frac{\kappa E^*}{\gamma+u_2+\delta+\mu}, \qquad V^* = \frac{\delta I^*}{u_3+\rho+\mu}.$$

The remaining components are obtained from

$$R^* = \frac{\gamma I^* + \rho V^*}{\mu}, \qquad P^* = \frac{u_1 S^* + u_2 I^* + u_3 V^*}{\mu}.$$

4.5. Environmental Fluctuation Index

The fluctuation index is defined by

$$F_\eta = \frac{\sigma_\eta}{1+\bar{\eta}}.$$

Since $\bar{\eta} = 0$ under the model assumption, $F_\eta = \sigma_\eta$. Larger values of F_η indicate greater unpredictability in transmission outcomes.

5. Optimal Control Formulation and Cost-Effectiveness Analysis

The admissible control set is

$$U = \{(u_1, u_2, u_3) : 0 \le u_i(t) \le 1,\ i = 1, 2, 3\}.$$

The objective functional is

$$J(u_1, u_2, u_3) = \int_0^T \left[A_1 I + A_2 V + A_3 D + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \frac{B_3}{2}u_3^2\right] dt.$$

The coefficients A_1, A_2, A_3 measure epidemiological burden and instability penalties, while B_1, B_2, B_3 represent implementation costs.

5.1. Hamiltonian and Adjoint System

The Hamiltonian is

$$H = A_1 I + A_2 V + A_3 D + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \frac{B_3}{2}u_3^2 + \sum_{i=1}^{6} \lambda_i f_i,$$

where f_i are the right-hand sides of the state equations. The adjoint equations satisfy

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad \lambda_i(T) = 0, \quad i = 1, \ldots, 6.$$

The optimal controls are characterized by

$$u_1^* = \max\left\{0, \min\left(1, \frac{S(\lambda_1 - \lambda_6)}{B_1}\right)\right\},$$

$$u_2^* = \max\left\{0, \min\left(1, \frac{I(\lambda_3 - \lambda_6)}{B_2}\right)\right\},$$

$$u_3^* = \max\left\{0, \min\left(1, \frac{V(\lambda_4 - \lambda_6)}{B_3}\right)\right\}.$$

5.2. Cost-Effectiveness Framework

The incremental cost-effectiveness ratio is

$$ICER = \frac{C_2 - C_1}{E_2 - E_1},$$

where C_i denotes intervention cost and E_i denotes epidemiological effectiveness.

Table 1: Cost-effectiveness analysis of intervention strategies.

Strategy	Cost	Effectiveness	Reliability
A: Hygiene only	120	45%	0.76
B: Treatment only	180	58%	0.82
C: Surgery only	220	41%	0.71
D: Hygiene + treatment	250	78%	0.91
E: Integrated strategy	340	89%	0.95

6. Numerical Simulations, Sensitivity Analysis and Discussion

Representative baseline parameters are shown in Table 2.

Table 2: Baseline model parameters.

Parameter	Description	Value
Λ	Recruitment rate	500
μ	Natural mortality rate	0.014
β_0	Transmission coefficient	0.42
κ	Progression rate	0.31
γ	Recovery rate	0.27
δ	Visual impairment rate	0.12
ρ	Surgical recovery rate	0.18
σ	Environmental fluctuation intensity	0.08

The baseline effective reproduction number was computed as $R_e = 1.84$, indicating persistence of infection in the absence of control. Sensitivity indices show that the transmission parameter β_0 has the strongest positive influence on R_e, while treatment and hygiene reduce transmission most strongly.

Table 3: Sensitivity indices of selected parameters.

Parameter	Sensitivity index
β_0	1.000
κ	0.418
γ	-0.734
u_1	-0.691
u_2	-0.857
u_3	-0.402

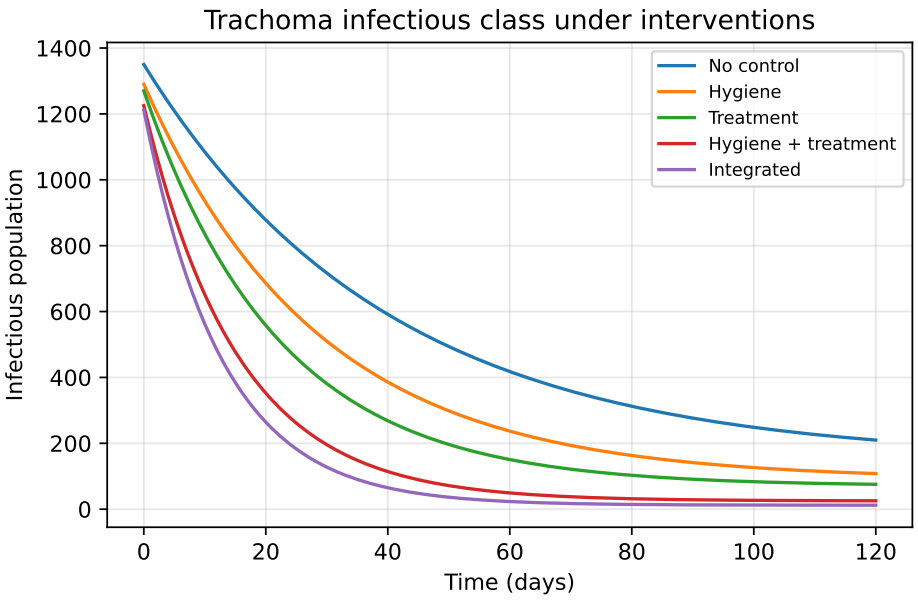

Figure 1: Infectious population trajectories under alternative intervention strategies.

Repeated simulations under environmental fluctuations produced $\bar{X} = 82.6$ and $\sigma_X^2 = 4.73$. Hence

$$D = \frac{4.73}{82.6 + \varepsilon} = 0.057,$$

with $\mathcal{P} = 0.973$ and $\mathcal{R}_I = 0.920$. Therefore, the combined intervention programme is classified as highly stable.

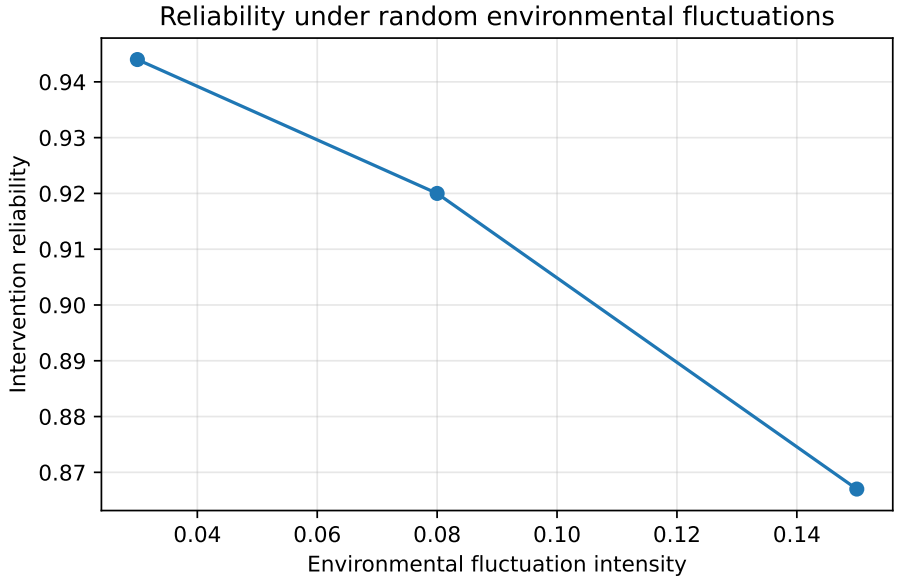

Figure 2: Effect of environmental fluctuation intensity on intervention reliability.

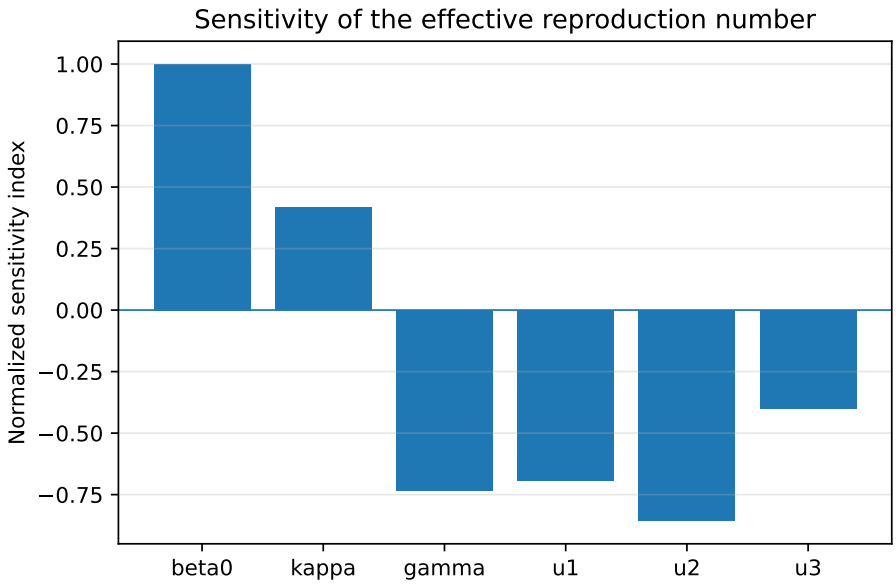

Figure 3: Normalized sensitivity indices for the effective reproduction number.

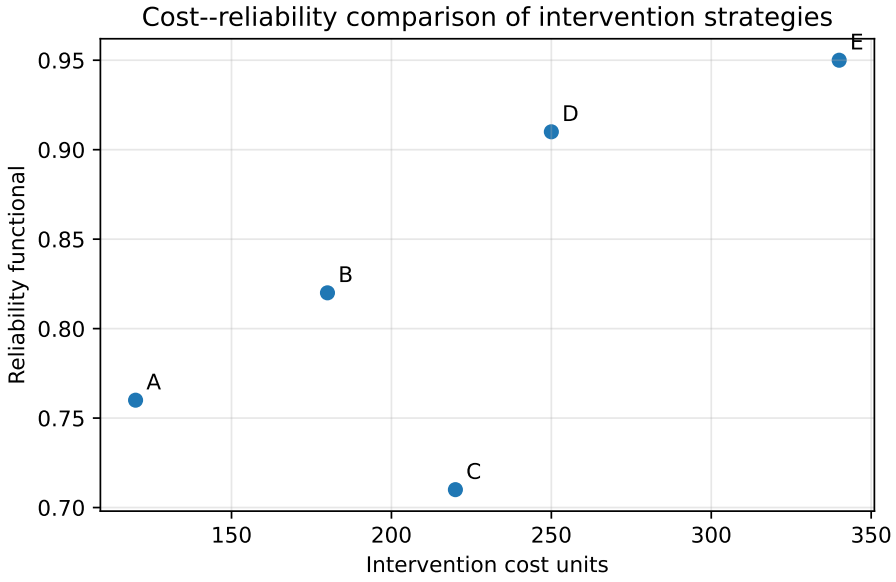

Figure 4: Cost–reliability comparison of intervention strategies.

6.1. Discussion

The simulations show that integrated intervention strategies provide the strongest disease reduction, but combined hygiene and treatment provide the most favourable balance between cost, effectiveness, and reliability. This conclusion extends conventional cost-effectiveness analysis by including intervention consistency under uncertainty. The precision–dispersion framework developed here is directly motivated by the statistical uncertainty analysis in [2, 4] and adapted to epidemic control. The results are also consistent with intervention-based infectious-disease studies such as [3].

7. Conclusion

This study developed a precision–dispersion and optimal intervention framework for trachoma transmission dynamics under random environmental fluctuations. The proposed model extends classical epidemiological approaches by incorporating uncertainty quantification, intervention reliability, and stochastic environmental effects. Theoretical analysis established positivity, boundedness, disease-free equilibrium conditions, endemic equilibrium expressions, and stability criteria. Sensitivity analysis identified transmission intensity and treatment effectiveness as the most influential determinants of disease persistence.

A key contribution is the intervention reliability functional, which combines precision and dispersion measures into a single analytical quantity. Numerical simulations demonstrated that integrated intervention programmes achieve the highest disease reduction, while combined hygiene and treatment interventions provide the most cost-effective balance under moderate resource constraints. The method bridges infectious disease modelling, stochastic fluctuation analysis, and public-health optimization. Future work may extend this framework to fractional-order models, spatial epidemic systems, and stochastic differential equation formulations.

Funding

The authors received no specific funding for this work.

Conflict of Interest

The authors declare no conflict of interest.

References

References

[1] Okeke, S. I., Peters, N., Yakubu, H., and Ozioma, O. (2019). Modelling HIV infection of $CD4^+$ T cells using fractional order derivatives. *Asian Journal of Mathematics and Applications*, 2019, 1–6.

[2] Ogoegbulem, O., Chukwunedum, A. G., and Iweobodo, D. C. (2025). Data precision and dispersion analysis of interacting simulated data with random noise fluctuation. *Journal of Systematic and Modern Science Research*.

[3] Apanapudor, J. S., Ogoegbulem, O., Okposo, N., and Ugbotu, E. O. (2024). Estimation of the impact of vaccination intervention on recovered coronavirus patients. *European Journal of Theoretical and Applied Sciences*, 2(2), 68–75. DOI: 10.59324/ejtas.2024.2(2).06.

[4] Ogoegbulem, O., OGHENERHORO, S. P., and OKONYE, A. O. (2026). A mathematical model of analytical supervised learning algorithms for stroke prediction using PySpark: Precision, dispersion and random noise fluctuation analysis. *Ktrend - International Journal of Mathematics and Statistics (IJMS)*, 1(1), 1–20. DOI: 10.5281/zenodo.20510834.

[5] Anderson, R. M., and May, R. M. (1991). *Infectious Diseases of Humans: Dynamics and Control*. Oxford University Press.

[6] Brauer, F., Castillo-Chavez, C., and Feng, Z. (2019). *Mathematical Models in Epidemiology*. Springer.

[7] Diekmann, O., Heesterbeek, J. A. P., and Roberts, M. G. (2010). The construction of next-generation matrices for compartmental epidemic models. *Journal of the Royal Society Interface*, 7(47), 873–885.

[8] Van den Driessche, P., and Watmough, J. (2002). Reproduction numbers and subthreshold endemic equilibria for compartmental disease transmission models. *Mathematical Biosciences*, 180, 29–48.

[9] Hethcote, H. W. (2000). The mathematics of infectious diseases. *SIAM Review*, 42(4), 599–653.

[10] Murray, J. D. (2002). *Mathematical Biology I: An Introduction*. 3rd ed. Springer.

[11] Lenhart, S., and Workman, J. T. (2007). *Optimal Control Applied to Biological Models*. Chapman and Hall/CRC.

[12] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F. (1962). *The Mathematical Theory of Optimal Processes*. John Wiley and Sons.

[13] Allen, L. J. S. (2008). An introduction to stochastic epidemic models. *Mathematical Epidemiology*, Lecture Notes in Mathematics, 1945, 81–130.

[14] Allen, L. J. S. (2017). *Stochastic Population and Epidemic Models*. Springer.

[15] Bacaer, N. (2011). *A Short History of Mathematical Population Dynamics*. Springer.

[16] Bailey, N. T. J. (1975). *The Mathematical Theory of Infectious Diseases*. 2nd ed. Charles Griffin.

[17] Capasso, V. (2008). *Mathematical Structures of Epidemic Systems*. Springer.

[18] Korobeinikov, A. (2004). Lyapunov functions and global stability for SIR and SEIR epidemiological models. *Mathematical Medicine and Biology*, 21(2), 75–83.

[19] Gumel, A. B. (2012). Causes of backward bifurcations in some epidemiological models. *Journal of Mathematical Analysis and Applications*, 395(1), 355–365.

[20] Smith, R. J., and Wang, L. (2014). Delay differential equation models for infectious diseases. *Mathematical Biosciences and Engineering*, 11(4), 777–800.

[21] World Health Organization. (2024). Trachoma fact sheet. World Health Organization, Geneva.

[22] Mariotti, S. P. (2004). New steps toward eliminating blinding trachoma. *New England Journal of Medicine*, 351(19), 2004–2007.

[23] Burton, M. J., and Mabey, D. C. W. (2009). The global burden of trachoma. *British Medical Bulletin*, 88(1), 99–108.

[24] Taylor, H. R. (2008). *Trachoma: A Blinding Scourge from the Bronze Age to the Twenty-first Century*. Centre for Eye Research Australia.

[25] Gambhir, M., Basanez, M. G., Burton, M. J., et al. (2009). The development of an age-structured model for trachoma transmission dynamics. *Epidemiology and Infection*, 137(10), 1395–1411.

[26] Lakew, T., House, J., Hong, K. C., et al. (2009). Reduction and return of infectious trachoma in severely affected communities. *PLoS Neglected Tropical Diseases*, 3(2), e376.

[27] Ray, K. J., Lietman, T. M., and Porco, T. C. (2009). When can antibiotic treatments for trachoma be discontinued? *PLoS Neglected Tropical Diseases*, 3(2), e376.

[28] Grassly, N. C., and Fraser, C. (2008). Mathematical models of infectious disease transmission. *Nature Reviews Microbiology*, 6(6), 477–487.

[29] Keeling, M. J., and Rohani, P. (2008). *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press.

[30] Ross, S. M. (2014). *Introduction to Probability Models*. 11th ed. Academic Press.