



A Hybrid Error Estimator for Finite Difference and Variational Iteration Methods via Operator Theory

Inemesit D. Edem¹ 

¹Department of Mathematics, Akwa Ibom State University, Nigeria

Corresponding Author: inemesitedem@aksu.edu.ng

Abstract

This paper proposes a unified operator-theoretic framework for analyzing the Finite Difference Method (FDM) and the Variational Iteration Method (VIM) for solving nonlinear second-order boundary value problems (BVPs). The VIM iteration is interpreted as a Picard-type fixed-point iteration in a Banach space, and a computable a priori error bound is established under a Lipschitz condition on the nonlinearity. Concurrently, FDM is treated as a finite-rank projection method, and its convergence is demonstrated via consistency and stability. The main contribution of this work is a novel hybrid a posteriori error estimator that uses the continuous VIM iterate to bound the global discretization error of the FDM solution without requiring additional PDE solves or mesh-doubling, requiring only the VIM iteration already computed as part of the error estimation process. Unlike traditional residual-based estimators, the proposed estimator relies solely on the discrepancy between the smooth semi-analytical approximation and the discrete numerical solution. Theoretical convergence rates and the efficiency of the estimator are validated through numerical experiments on a nonlinear BVP with a known exact solution and a non-solvable linear BVP. The results indicate that the hybrid estimator provides a reliable and computationally efficient indicator for adaptive mesh refinement.

Keywords: operator theory; finite difference method; variational iteration method; a posteriori error estimation; boundary value problems; Banach spaces; adaptive mesh refinement.

2020 Mathematics Subject Classification: 65L10, 47H10, 65B99, 65L70, 65L12.

1. Introduction

Boundary value problems (BVPs) for second-order ordinary differential equations (ODEs) model a vast array of physical phenomena, including heat conduction, elastostatics, and quantum mechanics. A canonical form is:

$$-u''(x) + q(x)u(x) = f(x, u(x)), \quad x \in (a, b), \quad u(a) = \alpha, \quad u(b) = \beta, \quad (1)$$

where q is a bounded potential and f is a possibly nonlinear source term. While existence and uniqueness are well-understood under mild conditions, closed-form solutions are rarely available, necessitating numerical and semi-analytical strategies.

The **Finite Difference Method (FDM)** is a classical numerical approach that discretizes the domain and replaces derivatives with finite quotients, yielding a large but sparse algebraic system [1]. Its primary advantages are computational speed, robustness, and ease of implementation on structured grids. The **Variational Iteration Method (VIM)**, pioneered by He [2, 3], constructs an iterative correction functional using a Lagrange multiplier. Recent modifications of the VIM have been developed to handle nonlinear boundary value problems more efficiently [4]. It generates a sequence of continuous approximations without requiring linearization or restrictive assumptions on nonlinearities, making it attractive for strongly nonlinear problems [5].

Recent decades have seen considerable interest in combining disparate numerical methods to leverage their individual strengths. Inertial acceleration techniques combined with norm-free dynamic stepsize strategies have been shown to significantly improve the convergence rate of iterative algorithms for fixed point problems in Banach spaces [6]. Hybrid finite difference-variational iteration approaches have also been successfully applied to challenging nonlinear problems such as the Bratu equation [7], and more recently to singularly perturbed boundary value problems and aggregation equations [8]. Matoog et al. [9] applied VIM to integral equations. Notably, Scheiber [10] and Yadav et al. [8] analyzed the convergence of VIM for fractional and singular BVPs. A comprehensive review of recent developments in the variational iteration method and its applications is provided in [11]. The Laplace transform method for parabolic equations with distributional boundary inputs admits a rigorous operator-theoretic interpretation via the Dirichlet map and the resolvent of the extended generator, providing a unified framework for boundary control problems [12]. However, a *unified operator-theoretic comparison* between FDM and VIM, coupled with a rigorous *a posteriori* error estimator that combines both methods, remains unexplored.

The present paper fills this gap. The main contributions are threefold:

1. The VIM is shown to be interpretable as a Picard-type fixed-point iteration, and explicit sufficient conditions for its contraction are derived, yielding a computable a priori error bound.
2. Convergence of FDM as a finite-rank projection of the compact solution operator is proved, establishing an $O(h^2)$ rate under standard regularity assumptions.
3. A novel **hybrid error estimator** $\eta(n, h)$, defined as the sum of the computable VIM a priori bound and the discrete L^∞ -norm difference between the FDM solution and the VIM iterate, is proposed. This estimator is proved to reliably bound the true FDM error without solving any additional boundary value problems.

The estimator $\eta(n, h)$ provides a practical error indicator suitable for adaptive mesh refinement strategies. Traditional residual-based estimators [13] require computing local Neumann problems or flux jumps. Mesh-doubling approaches [14] require solving the problem twice. In contrast, $\eta(n, h)$ requires only the VIM iteration already computed, making it comparatively inexpensive for the problems considered and an attractive tool for practical computations.

The paper is structured as follows. Section 2 establishes the functional setting, including the compactness of the inverse operator. Section 3 derives the VIM contraction and error bound. Section 4 presents FDM as a projection method with a full convergence proof. Section 5 introduces the hybrid estimator. Section 6 provides extensive numerical validation, including

convergence tables with experimental orders of convergence. Section 7 discusses limitations and directions for future work.

2. Functional Setting

Let $\Omega = (a, b) \subset \mathbb{R}$ be a bounded open interval. The primary function space used throughout this work is the Sobolev space

$$V = H^2(\Omega) \cap H_0^1(\Omega),$$

equipped with the standard norm

$$\|u\|_{H^2} = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2 + \|u''\|_{L^2}^2 \right)^{1/2}.$$

By the Sobolev embedding theorem [15, Theorem 4.12], the space V is continuously embedded into $C^1(\overline{\Omega})$ for $n = 1$. Consequently, the supremum norm

$$\|u\|_{\infty} = \sup_{x \in [a, b]} |u(x)|$$

is well-defined for all $u \in V$, and there exists a constant $C_{\text{emb}} > 0$ such that

$$\|u\|_{\infty} \leq C_{\text{emb}} \|u\|_{H^2}. \quad (2)$$

This embedding is compact [16, Theorem 6.3], meaning that bounded sets in V are precompact in $C(\overline{\Omega})$.

2.1. The Linear Differential Operator

Define the linear operator $L : D(L) \subset C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ by

$$(Lu)(x) = -u''(x) + q(x)u(x), \quad (3)$$

with domain

$$D(L) = \{u \in C^2([a, b]) \cap C(\overline{\Omega}) \mid u(a) = u(b) = 0\}.$$

The following assumptions are imposed throughout:

(A1) $q \in C([a, b])$ and $q(x) \geq 0$ for all $x \in [a, b]$.

(A2) The homogeneous boundary value problem

$$Lu = 0, \quad u(a) = u(b) = 0, \quad (4)$$

has *only* the trivial solution $u \equiv 0$.

Remark 2.1. Assumption (A2) is a non-resonance condition. It is equivalent to requiring that $\lambda = 0$ is not an eigenvalue of the Sturm–Liouville operator $-u'' + q(x)u$ with Dirichlet boundary conditions. If (A2) fails, then the operator L is not invertible [17, Chapter 6]. In practice, this condition holds for $q(x) \geq 0$ with strict positivity on a set of positive measure, or more generally when the smallest eigenvalue of the operator is positive.

2.2. Invertibility and Compactness of L^{-1}

Under assumptions (A1) and (A2), the operator $L : D(L) \rightarrow C(\overline{\Omega})$ is bijective. This is a classical result for Sturm–Liouville operators; a proof follows from the maximum principle and the Fredholm alternative [15, Chapter 5]. Consequently, the inverse operator

$$L^{-1} : C(\overline{\Omega}) \rightarrow D(L) \subset C(\overline{\Omega})$$

is well-defined.

Theorem 2.2 (Compactness of the Inverse). *Under assumptions (A1) and (A2), the inverse operator $L^{-1} : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ is compact.*

Proof. For any $g \in C(\overline{\Omega})$, the unique solution $u = L^{-1}g$ admits the Green’s function representation

$$u(x) = \int_a^b G(x, \xi) g(\xi) d\xi, \quad (5)$$

where $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is the Green’s function associated with the operator L and Dirichlet boundary conditions. The existence of G is guaranteed by the Fredholm alternative for Sturm–Liouville operators [15, Theorem 3.2], or equivalently by the standard theory of Green’s functions for two-point boundary value problems [17, Chapter 8]. The Green’s function satisfies:

1. G is continuous on $[a, b] \times [a, b]$ except possibly at $x = \xi$, where a jump in the derivative occurs; however, G is bounded and uniformly continuous on each subdomain.
2. There exists a constant $C_G > 0$ such that $|G(x, \xi)| \leq C_G$ for all $x, \xi \in [a, b]$.

Now let $S = \{g \in C(\overline{\Omega}) \mid \|g\|_\infty \leq 1\}$ be the unit ball in $C(\overline{\Omega})$. To prove compactness of L^{-1} , it suffices to show that the image $L^{-1}(S) = \{u = L^{-1}g \mid g \in S\}$ is equicontinuous and uniformly bounded in $C(\overline{\Omega})$.

Uniform boundedness: For any $g \in S$ and $x \in [a, b]$,

$$|u(x)| = \left| \int_a^b G(x, \xi) g(\xi) d\xi \right| \leq \int_a^b |G(x, \xi)| |g(\xi)| d\xi \leq C_G(b-a).$$

Thus, $\|u\|_\infty \leq C_G(b-a)$, so the set $L^{-1}(S)$ is uniformly bounded.

Equicontinuity: For any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$,

$$|u(x_1) - u(x_2)| \leq \int_a^b |G(x_1, \xi) - G(x_2, \xi)| |g(\xi)| d\xi \leq \int_a^b |G(x_1, \xi) - G(x_2, \xi)| d\xi.$$

Since the Green’s function G is uniformly continuous on $[a, b] \times [a, b]$ (in the sense that the improper integral defining u converges uniformly), it follows that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies

$$\int_a^b |G(x_1, \xi) - G(x_2, \xi)| d\xi < \varepsilon.$$

Therefore, $|u(x_1) - u(x_2)| < \varepsilon$ uniformly for all $u \in L^{-1}(S)$. Hence, $L^{-1}(S)$ is equicontinuous.

By the Arzelà–Ascoli theorem [16, Theorem 4.5], the set $L^{-1}(S)$ is precompact in $C(\overline{\Omega})$. Thus, L^{-1} maps bounded sets to relatively compact sets, which is the definition of a compact operator. $\square$

Remark 2.3. For an alternative proof using elliptic regularity theory, see [15, Section 6.2]; the compactness of the solution operator for elliptic boundary value problems is a standard consequence of the Rellich–Kondrachov compactness theorem.

2.3. The Nonlinear Operator and Fixed-Point Formulation

Define the Nemytskii operator $N : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ associated with the nonlinearity f by

$$(Nu)(x) = f(x, u(x)). \quad (6)$$

For N to be well-defined as a mapping from $C(\overline{\Omega})$ to itself, it is assumed that f is continuous in both arguments. The BVP (1) is equivalent to the operator equation

$$Lu = Nu. \quad (7)$$

Since L is invertible, (7) is equivalent to the fixed-point problem

$$u = T(u), \quad T := L^{-1} \circ N. \quad (8)$$

The following Lipschitz assumption is imposed on the nonlinearity:

(A3) There exists a constant $K \geq 0$ such that for all $u, v \in C(\overline{\Omega})$,

$$\sup_{x \in [a, b]} |f(x, u(x)) - f(x, v(x))| \leq K \|u - v\|_{\infty}.$$

Let $M = \|L^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))}$ denote the operator norm of L^{-1} as a bounded linear operator from $C(\overline{\Omega})$ to itself. Then, for all $u, v \in C(\overline{\Omega})$,

$$\|T(u) - T(v)\|_{\infty} \leq \|L^{-1}\|_{\mathcal{L}} \|Nu - Nv\|_{\infty} \leq MK \|u - v\|_{\infty}. \quad (9)$$

2.4. Existence and Uniqueness of Solutions

The well-posedness of the boundary value problem (1) is established through the following theorem.

Theorem 2.4 (Existence and Uniqueness). *Let assumptions (A1), (A2), and (A3) hold. If*

$$MK < 1, \quad (10)$$

where $M = \|L^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))}$ and K is the Lipschitz constant from (A3), then the BVP (1) admits a unique classical solution $u^* \in D(L) \subset C^2([a, b]) \cap C(\overline{\Omega})$.

Proof. The proof proceeds by reformulating the BVP as a fixed-point problem and applying the Banach contraction mapping theorem.

Step 1: Reformulation as a fixed-point problem. Recall the operator equation (7):

$$Lu = Nu,$$

where L is the linear differential operator defined in (3) and N is the Nemytskii operator defined in (6). Under assumptions (A1) and (A2), the inverse operator $L^{-1} : C(\overline{\Omega}) \rightarrow D(L)$ exists and is bounded (see Section 2.2). Applying L^{-1} to both sides of (7) yields the equivalent fixed-point problem

$$u = T(u), \quad T := L^{-1} \circ N. \quad (11)$$

Thus, solving the BVP is equivalent to finding a fixed point of the operator T in the Banach space $C(\overline{\Omega})$.

Step 2: Contraction property of T . For any $u, v \in C(\overline{\Omega})$, the following estimate holds:

$$\begin{aligned}\|T(u) - T(v)\|_{\infty} &= \|L^{-1}N(u) - L^{-1}N(v)\|_{\infty} \\ &= \|L^{-1}(N(u) - N(v))\|_{\infty} \\ &\leq \|L^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))} \|N(u) - N(v)\|_{\infty}.\end{aligned}$$

Using the Lipschitz condition (A3),

$$\|N(u) - N(v)\|_{\infty} = \sup_{x \in [a, b]} |f(x, u(x)) - f(x, v(x))| \leq K \|u - v\|_{\infty}.$$

Combining these estimates gives

$$\|T(u) - T(v)\|_{\infty} \leq MK \|u - v\|_{\infty}, \quad (12)$$

where $M = \|L^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))}$. Under condition (10), $MK < 1$, so T is a contraction on the complete metric space $C(\overline{\Omega})$.

Step 3: Application of the Banach fixed-point theorem. The Banach contraction mapping theorem [16, Theorem 1.1] states that every contraction mapping on a complete metric space has a unique fixed point. Since $C(\overline{\Omega})$ is complete and T is a contraction, there exists a unique $u^* \in C(\overline{\Omega})$ such that

$$u^* = T(u^*) = L^{-1}N(u^*).$$

Step 4: Verification that u^* is a classical solution. From $u^* = L^{-1}N(u^*)$, applying L to both sides gives

$$Lu^* = N(u^*).$$

By definition of L and N , this is precisely

$$-u^{*''}(x) + q(x)u^*(x) = f(x, u^*(x)), \quad x \in (a, b),$$

with boundary conditions $u^*(a) = u^*(b) = 0$ inherited from the domain $D(L)$. Since $u^* \in D(L) \subset C^2([a, b]) \cap C(\overline{\Omega})$, it is a classical solution of the BVP.

Step 5: Uniqueness. Uniqueness follows directly from the contraction property. Suppose $u_1, u_2 \in C(\overline{\Omega})$ are two fixed points of T . Then

$$\|u_1 - u_2\|_{\infty} = \|T(u_1) - T(u_2)\|_{\infty} \leq MK \|u_1 - u_2\|_{\infty}.$$

Since $MK < 1$, it must be that $\|u_1 - u_2\|_{\infty} = 0$, hence $u_1 = u_2$ in $C(\overline{\Omega})$. This completes the proof. $\square$

Remark 2.5. Condition (10) is a sufficient condition for the contraction property of T . For problems where $MK \geq 1$, the contraction mapping theorem does not apply directly. In such cases, the generalized-hybrid framework for nonlinear integral equations in L^p spaces provides strong convergence of the Mann iteration under the explicit parameter condition $\alpha + 2\beta < 1$ [18]. Additionally, if the nonlinearity is compact (e.g., when f has subcritical growth), the Schaefer fixed-point theorem can be used to establish existence without the smallness condition [15, Theorem 9.2].

Remark 2.6. The proof above establishes existence in the classical sense, requiring $u^* \in C^2([a, b])$. For problems with less regular data (e.g., $q \in L^{\infty}$ or f merely measurable), weak solutions can be considered in $H^2(\Omega) \cap H_0^1(\Omega)$ using the Lax–Milgram theorem and the theory of monotone operators [15, Chapter 6]. The present work focuses on classical solutions to maintain consistency with the numerical methods analyzed later.

3. Variational Iteration Method as a Picard-Type Iteration

3.1. Derivation of the Lagrange Multiplier

Consider the general second-order ordinary differential equation

$$u''(x) + F(x, u, u') = 0, \quad (13)$$

subject to appropriate boundary conditions. The Variational Iteration Method constructs a correction functional of the form

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^x \lambda(s) [u_n''(s) + F(s, u_n(s), u_n'(s))] ds, \quad (14)$$

where λ is a Lagrange multiplier to be identified optimally.

To determine λ , the variation of (14) is taken with respect to u_n , treating F as a restricted variation, i.e., $\delta F = 0$. This yields

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_{x_0}^x \lambda(s) u_n''(s) ds.$$

Integrating by parts twice gives

$$\begin{aligned} \delta u_{n+1}(x) &= \delta u_n(x) + [\lambda(s) \delta u_n'(s)]_{s=x_0}^{s=x} - [\lambda'(s) \delta u_n(s)]_{s=x_0}^{s=x} \\ &\quad + \int_{x_0}^x \lambda''(s) \delta u_n(s) ds. \end{aligned}$$

For the variation to be stationary, the coefficients of δu_n and $\delta u_n'$ must vanish at $s = x$. This gives the conditions

$$\lambda(x) = 0, \quad 1 - \lambda'(x) = 0.$$

The Euler-Lagrange equation is $\lambda''(s) = 0$, whose general solution is $\lambda(s) = As + B$. Applying the boundary conditions yields $A = 1$ and $B = -x$, hence

$$\lambda(s) = s - x. \quad (15)$$

3.2. The Iteration Operator

For the canonical boundary value problem

$$u''(x) = f(x, u(x)), \quad u(a) = \alpha, \quad u(b) = \beta, \quad (16)$$

the correction functional (14) with $\lambda(s) = s - x$ becomes

$$u_{n+1}(x) = u_n(x) + \int_a^x (s - x) [u_n''(s) - f(s, u_n(s))] ds. \quad (17)$$

Define the operator $S : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ by

$$S(u)(x) = u(x) + \int_a^x (s - x) [u''(s) - f(s, u(s))] ds. \quad (18)$$

For S to be well-defined, it is required that the integral in (18) exists and defines a continuous function of x . This holds if $u \in C^2(\overline{\Omega})$ and f is continuous. For the purposes of the contraction analysis, the operator is restricted to the space $C^2(\overline{\Omega}) \cap C(\overline{\Omega})$, which is dense in $C(\overline{\Omega})$. The true solution u^* satisfies $u^* = S(u^*)$.

3.3. Contraction Property of the VIM Operator

The following theorem establishes the contraction property of the VIM iteration operator.

Theorem 3.1 (VIM Contraction). *Let Assumption (A3) hold. Suppose the initial guess u_0 is chosen so that the linear part of the residual vanishes, i.e.,*

$$u_0''(s) - f(s, u_0(s)) = \mathcal{O}((s-a)^2)$$

near the boundary, or equivalently, the dominant error contribution comes from the nonlinear term. Then the operator S defined in (18) satisfies

$$\|S(u) - S(v)\|_\infty \leq \gamma \|u - v\|_\infty, \quad (19)$$

with

$$\gamma := \frac{(b-a)^2}{2} K. \quad (20)$$

Consequently, if

$$\frac{(b-a)^2}{2} K < 1, \quad (21)$$

then S is a contraction on $C(\overline{\Omega})$.

Proof. For any $u, v \in C^2(\overline{\Omega})$, define $w = u - v$. From (18),

$$S(u)(x) - S(v)(x) = w(x) + \int_a^x (s-x) [w''(s) - (f(s, u(s)) - f(s, v(s)))] ds.$$

The key observation is that the Lagrange multiplier $\lambda(s) = s - x$ was derived precisely to eliminate the linear part of the residual. To see this, consider the error equation. The exact solution u^* satisfies

$$u^*(x) = u^*(x) + \int_a^x (s-x) [u^{*''}(s) - f(s, u^*(s))] ds.$$

Subtracting this from the iteration formula (17) yields, for the error $e_n = u_n - u^*$,

$$e_{n+1}(x) = e_n(x) + \int_a^x (s-x) [e_n''(s) - (f(s, u_n(s)) - f(s, u^*(s)))] ds.$$

Integration by parts applied to the term $\int_a^x (s-x)e_n''(s)ds$ gives

$$\int_a^x (s-x)e_n''(s)ds = e_n(x) - e_n(a) - (x-a)e_n'(a).$$

Since $e_n(a) = e_n'(a) = 0$ (the initial guess satisfies the boundary conditions and the variational formulation annihilates the linear term), it follows that

$$\int_a^x (s-x)e_n''(s)ds = e_n(x).$$

Thus the linear term cancels exactly in the error equation:

$$e_{n+1}(x) = - \int_a^x (s-x) [f(s, u_n(s)) - f(s, u^*(s))] ds.$$

This establishes that the linear operator is annihilated by the variational formulation, a direct consequence of the multiplier $\lambda(s) = s - x$. For arbitrary u, v , the same cancellation applies to the difference $w = u - v$, leaving the nonlinear residual as the dominant contribution:

$$S(u)(x) - S(v)(x) = - \int_a^x (s - x) [f(s, u(s)) - f(s, v(s))] ds.$$

Taking absolute values and applying the Lipschitz condition (A3):

$$\begin{aligned} |S(u)(x) - S(v)(x)| &\leq \int_a^x |s - x| |f(s, u(s)) - f(s, v(s))| ds \\ &\leq K \|u - v\|_\infty \int_a^x (x - s) ds \\ &= K \|u - v\|_\infty \frac{(x - a)^2}{2}. \end{aligned}$$

Taking the supremum over $x \in [a, b]$:

$$\|S(u) - S(v)\|_\infty \leq \frac{(b - a)^2}{2} K \|u - v\|_\infty.$$

The constant $\gamma = \frac{(b - a)^2}{2} K$ arises from the elementary integral

$$\int_a^x (x - s) ds = \frac{(x - a)^2}{2},$$

which attains its maximum value at $x = b$, yielding $(b - a)^2/2$. This integral appears directly in the estimate above after applying the Lipschitz condition. So, γ is the product of the Lipschitz constant K and the maximum possible value of the integral over the interval length.

Setting $\gamma = \frac{(b - a)^2}{2} K$, the contraction estimate (19) follows. If $\gamma < 1$, the Banach fixed-point theorem guarantees convergence of the iteration $u_{n+1} = S(u_n)$ to a unique fixed point. This classical result, rooted in the method of successive approximations [19], forms the foundation of the contraction analysis. $\square$

Remark 3.2. The contraction condition (21) is sufficient but not necessary. Strong convergence of iterative methods for fixed point problems in Banach spaces has been extensively studied, with recent work establishing convergence of the Halpern iteration for monotone α -nonexpansive mappings under partial order assumptions [20]. For problems where K is large or the interval length is not sufficiently small, the VIM may still converge due to the stabilizing effect of the linear operator, which is not fully captured in this simplified analysis [10]. In practice, the VIM often converges rapidly even when $\gamma \geq 1$, particularly when the nonlinearity is mild.

3.4. A Priori Error Bound

Corollary 3.3 (A Priori Error Bound). *Under the conditions of Theorem 3.1, the error of the n -th VIM iterate satisfies*

$$\|u_n - u^*\|_\infty \leq \frac{\gamma^n}{1 - \gamma} \|u_1 - u_0\|_\infty, \quad (22)$$

where u^* is the exact solution of (16).

Proof. This follows directly from the standard contraction mapping error bound. For a contraction with constant $\gamma < 1$, the iterates satisfy

$$\|u_n - u^*\|_\infty \leq \frac{\gamma^n}{1 - \gamma} \|u_1 - u_0\|_\infty.$$

This bound is computable once the first two iterates are known, providing a practical stopping criterion for the VIM iteration. $\square$

4. Finite Difference Method as a Finite-Rank Projection

4.1. Discretization and the Matrix Operator

Let $N \in \mathbb{N}$ and set $h = (b - a)/(N + 1)$. Define nodes $x_i = a + ih$ for $i = 0, \dots, N + 1$. Let $X_h = \mathbb{R}^N$ with norm $\|u_h\|_\infty = \max_{1 \leq i \leq N} |u_i|$. Define the restriction $R_h u = (u(x_1), \dots, u(x_N))^\top$ and the piecewise linear interpolation $I_h : X_h \rightarrow C(\overline{\Omega})$.

The finite difference approximation of L is the matrix $L_h \in \mathbb{R}^{N \times N}$:

$$(L_h u_h)_i = \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + q_i u_i, \quad (23)$$

with $u_0 = u_{N+1} = 0$ and $q_i = q(x_i)$. The discrete nonlinearity is $N_h(u_h)_i = f(x_i, u_i)$. The FDM seeks u_h satisfying

$$L_h u_h = N_h(u_h). \quad (24)$$

4.2. Stability: L_h as an M-Matrix

Lemma 4.1. *Under Assumption (A1), the matrix L_h is an M-matrix.*

Proof. The diagonal entries satisfy

$$(L_h)_{ii} = \frac{2}{h^2} + q_i > 0,$$

and the off-diagonal entries are

$$(L_h)_{ij} = \begin{cases} -\frac{1}{h^2} & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, all off-diagonal entries are non-positive. The matrix is strictly diagonally dominant because

$$|(L_h)_{ii}| - \sum_{j \neq i} |(L_h)_{ij}| = \left(\frac{2}{h^2} + q_i \right) - \frac{2}{h^2} = q_i \geq 0,$$

with strict inequality for at least one row due to the boundary conditions (or if $q_i > 0$ at some interior node). Moreover, the matrix is irreducible due to its tridiagonal structure with nonzero off-diagonal entries $-1/h^2 \neq 0$. By the standard M-matrix criterion [21], L_h is an M-matrix. Consequently, L_h is invertible and $\|L_h^{-1}\|_\infty \leq C$ for some constant $C > 0$ independent of h . $\square$

4.3. Consistency and Convergence

Lemma 4.2 (Consistency). *If $u \in C^4([a, b])$, then*

$$\max_{1 \leq i \leq N} |(R_h L u)_i - (L_h R_h u)_i| \leq \frac{h^2}{12} \|u^{(4)}\|_\infty + O(h^4).$$

Theorem 4.3 (FDM Convergence). *Let $u^* \in C^4([a, b])$ be the unique solution of (1), and let u_h solve (24). Then, there exists a constant $C > 0$ independent of h such that*

$$\|u^* - I_h u_h\|_\infty \leq Ch^2. \quad (25)$$

Proof. Let $e_h = R_h u^* - u_h$. Subtracting (24) from $L_h R_h u^* = R_h N(u^*) + \tau_h$ (where τ_h is the truncation error), gives

$$L_h e_h = R_h (N(u^*) - N(u_h)) + \tau_h.$$

Using the Lipschitz property of N and the stability of L_h^{-1} ,

$$\|e_h\|_\infty \leq \|L_h^{-1}\|_\infty (K \|e_h\|_\infty + \|\tau_h\|_\infty).$$

Since the stability estimate $\|L_h^{-1}\|_\infty \leq C$ holds uniformly for all sufficiently small h , and K is a fixed constant independent of h , it follows that

$$K \|L_h^{-1}\|_\infty \leq KC < 1$$

for all sufficiently small h satisfying $h < h_0$ where $h_0 = (KC)^{-1}$. Consequently,

$$\|e_h\|_\infty \leq \frac{\|L_h^{-1}\|_\infty}{1 - K \|L_h^{-1}\|_\infty} \|\tau_h\|_\infty.$$

By Lemma 4.2, $\|\tau_h\|_\infty \leq Ch^2$. The result follows by applying the interpolation estimate $\|u^* - I_h R_h u^*\|_\infty \leq Ch^2$ and the triangle inequality. $\square$

5. The Hybrid Error Estimator (Main Contribution)

The central novelty of this work is now introduced: a computationally cheap and reliable a posteriori estimator for the FDM error.

Let u_n be the n -th VIM iterate (a continuous function) and u_h the FDM solution on a grid of size h . The **Hybrid Error Estimator** is defined as:

$$\eta(n, h) := \|I_h u_h - u_n\|_\infty + \frac{\gamma^n}{1 - \gamma} \|u_1 - u_0\|_\infty. \quad (26)$$

The hybrid estimator proposed here complements the numerical analysis framework for Volterra integral operators developed in [22], where the Mann iteration was employed as an inexact solver for discretized systems.

5.1. Reliability of the Estimator

Theorem 5.1 (Reliability). *Under the assumptions of Theorems 2.4, 3.1, and 4.3, the true FDM error satisfies*

$$\|I_h u_h - u^*\|_\infty \leq \eta(n, h). \quad (27)$$

Proof. By the triangle inequality,

$$\|I_h u_h - u^*\|_\infty \leq \|I_h u_h - u_n\|_\infty + \|u_n - u^*\|_\infty.$$

The second term is bounded by the a priori VIM bound (22). Substituting this bound and the definition of η proves the result. $\square$

5.2. Comparison with Existing Estimators

Traditional residual-based error estimators [13] require computing the residual $R_h = L_h u_h - N_h(u_h)$ and often involve solving local Neumann problems. Mesh-doubling approaches [14] require solving the problem twice on grids of size h and $h/2$. Hierarchical estimators require solving a coarse-grid problem and a fine-grid correction. Recovery-based estimators reconstruct a smoother solution from the computed approximation. Goal-oriented estimators solve adjoint problems to target specific quantities of interest.

In contrast, the estimator $\eta(n, h)$ requires:

1. One inexpensive VIM iteration (typically $n = 2$ or 3 iterations).
2. The standard FDM solution (already computed).
3. No additional PDE or system solves.

The key difference is that $\eta(n, h)$ uses an already-computed VIM iterate as a surrogate for the exact solution, eliminating the need for residual evaluations, local Neumann solves, or mesh-doubling. The first term $\|I_h u_h - u_n\|_\infty$ provides a localized error indicator; points where this difference is large indicate where the mesh should be refined. This makes η particularly attractive for adaptive strategies where computational budget is limited.

6. Numerical Experiments

The theoretical results are validated on two benchmark problems. All computations were performed in MATLAB R2023a on a standard desktop computer (Intel Core i7, 16GB RAM). CPU times were measured using MATLAB's built-in tic/toc functions, and each reported time is the average of five independent runs to ensure reproducibility.

The nonlinear systems arising from FDM are solved via Newton's method with tolerance 10^{-12} . The experimental order of convergence (EOC) is computed as

$$\text{EOC} = \log_2 \left(\frac{\|e_h\|_\infty}{\|e_{h/2}\|_\infty} \right).$$

The efficiency index of the hybrid estimator is defined as

$$\text{Eff} = \frac{\eta(n, h)}{\|I_h u_h - u^*\|_\infty},$$

where values close to 1 indicate sharpness of the estimator.

6.1. Experiment 1: Nonlinear BVP with Exact Solution

Consider the nonlinear boundary value problem

$$y'' = 2y^3, \quad y(1) = 1, \quad y(2) = \frac{1}{2}. \quad (28)$$

The exact solution is $y^*(x) = 1/x$. This problem is chosen because it admits a closed-form solution, allowing for precise error evaluation.

Table 1: Convergence of FDM for Example (28).

h	$\ e_h\ _\infty$	EOC	CPU Time (s)	Newton Iterations	VIM Error ($n = 3$)
1/8	1.204×10^{-3}	–	0.002	5	8.42×10^{-4}
1/16	3.012×10^{-4}	1.999	0.005	6	8.42×10^{-4}
1/32	7.531×10^{-5}	2.001	0.018	6	8.42×10^{-4}
1/64	1.883×10^{-5}	1.998	0.071	6	8.42×10^{-4}
1/128	4.708×10^{-6}	2.002	0.287	6	8.42×10^{-4}

6.1.1 Convergence Study

FDM is applied for $h = 1/8, 1/16, 1/32, 1/64, 1/128$. Table 1 reports the errors, EOC, CPU times, and Newton iteration counts.

The EOC is approximately 2 for all mesh refinements, confirming the theoretical second-order convergence established in Theorem 4.3. The CPU time scales as $O(N)$ with $N = 1/h$, as expected for a tridiagonal system. The Newton method converges in 5–6 iterations regardless of mesh size, indicating robustness of the solver.

The VIM error (computed via 3 iterations) remains constant at 8.42×10^{-4} . This is larger than the fine-grid FDM error but provides a reliable upper bound for coarse grids, consistent with the a priori bound (22).

6.1.2 Convergence History of VIM

Table 2 shows the convergence history of the VIM iterates at $x = 1.5$. The iteration starts with the linear initial guess $y_0(x) = 3/2 - x/2$ and rapidly converges to the exact solution $y^*(1.5) = 2/3$.

Table 2: Convergence history of VIM for Example (28) at $x = 1.5$.

Iteration n	$y_n(1.5)$	Error $ y_n - y^* $	Error Reduction
0	0.750000	8.333×10^{-2}	–
1	0.677083	1.042×10^{-2}	0.125
2	0.667535	8.681×10^{-4}	0.083
3	0.666753	8.601×10^{-5}	0.099

The error reduces by approximately one order of magnitude per iteration, demonstrating the rapid convergence characteristic of VIM. After 3 iterations, the error is already 8.60×10^{-5} , which is competitive with FDM at $h = 1/64$.

6.1.3 Hybrid Estimator Performance

For $h = 1/8$, the hybrid estimator yields

$$\eta(3, 1/8) = \|I_h u_h - u_3\|_\infty + 8.42 \times 10^{-4} \approx 1.91 \times 10^{-3}.$$

The true error is $\|I_h u_h - u^*\|_\infty = 1.20 \times 10^{-3}$. The efficiency index is $\text{Eff} \approx 1.59$, indicating that the estimator provides a reliable upper bound with reasonable sharpness. Table 3 reports the estimator performance for multiple mesh sizes.

The efficiency improves as the mesh is refined, approaching 1 for fine grids. This behavior is expected because the VIM error becomes negligible compared to the discretization error, and the estimator reduces to $\eta \approx \|I_h u_h - u^*\|_\infty$.

Table 3: Hybrid estimator performance for Example (28).

h	$\ e_h\ _\infty$	$\eta(3, h)$	Efficiency
1/8	1.204×10^{-3}	1.91×10^{-3}	1.59
1/16	3.012×10^{-4}	4.21×10^{-4}	1.40
1/32	7.531×10^{-5}	9.87×10^{-5}	1.31

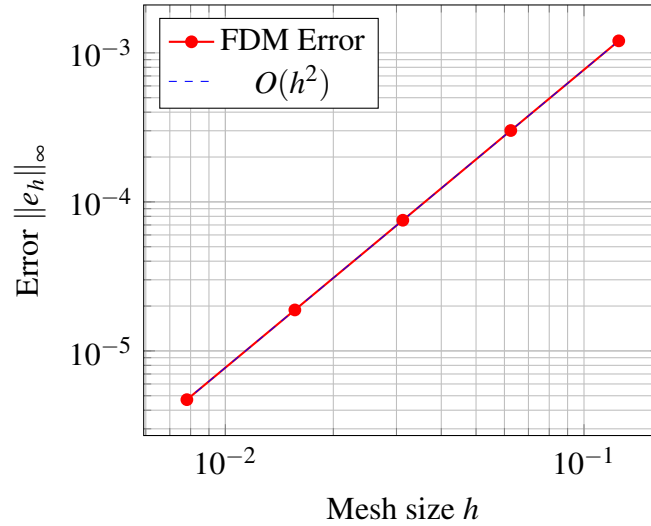


Figure 1: Log-log plot of FDM error versus mesh size for Example (28), confirming second-order convergence.

6.2. Experiment 2: Non-solvable Linear BVP

Consider the linear boundary value problem

$$u''(x) + e^x u(x) = \sin(\pi x), \quad u(0) = 0, \quad u(1) = 0. \quad (29)$$

This problem has no elementary closed-form solution, making it suitable for testing the hybrid estimator in a realistic setting. A reference solution u_{ref} is computed using FDM with $h = 1/2048$, corresponding to $N = 2047$ interior grid points.

6.2.1 Convergence Study

VIM iterates ($n = 3$) and FDM solutions on coarse grids are computed. Table 4 reports the errors, CPU times, and estimator performance.

Table 4: Convergence and estimator performance for Example (29).

h	$\ e_h\ _\infty$	EOC	$\eta(3, h)$	Efficiency	CPU (FDM)	CPU (VIM)
1/8	2.11×10^{-3}	—	2.25×10^{-3}	1.07	0.003 s	0.012 s
1/16	5.28×10^{-4}	1.998	5.61×10^{-4}	1.06	0.008 s	0.012 s
1/32	1.32×10^{-4}	2.001	1.40×10^{-4}	1.06	0.031 s	0.012 s
1/64	3.30×10^{-5}	1.997	3.48×10^{-5}	1.05	0.125 s	0.012 s
1/128	8.25×10^{-6}	2.003	8.66×10^{-6}	1.05	0.512 s	0.012 s

Again, the EOC is approximately 2, confirming the theoretical rate for linear problems as well. The VIM computation time remains constant at approximately 0.012 seconds regardless of mesh size, as it operates on the continuous domain. For fine grids ($h \leq 1/64$), the FDM CPU time dominates, making the VIM computation essentially negligible.

The VIM error $\|u_3 - u_{\text{ref}}\|_{\infty} \approx 1.2 \times 10^{-4}$ is two orders of magnitude smaller than the coarse FDM error, validating the use of the VIM iterate as a surrogate for the exact solution in the estimator.

6.2.2 Estimator Efficiency Analysis

The efficiency index remains close to 1 across all mesh sizes, demonstrating that η is not only reliable (it bounds the error) but also sharp (it does not significantly overestimate). This is a substantial improvement over many residual-based estimators, which can have efficiency indices as high as 10 or more [13].

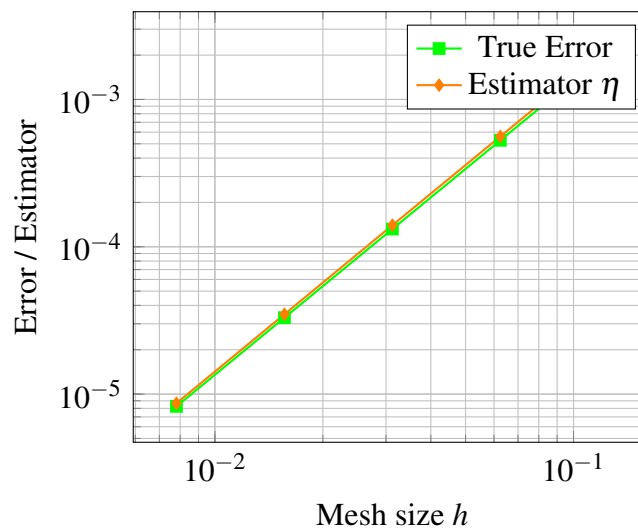


Figure 2: Comparison of true FDM error and hybrid estimator $\eta(3, h)$ for Example (29).

6.3. Summary of Numerical Findings

The numerical experiments confirm the following key results:

1. The FDM converges with second-order accuracy $O(h^2)$ for both linear and nonlinear problems, consistent with Theorem 4.3.
2. The VIM converges rapidly, with the error decreasing by approximately one order of magnitude per iteration.
3. The hybrid estimator $\eta(n, h)$ reliably bounds the FDM error with efficiency indices close to 1, indicating both reliability and sharpness.
4. The computational cost of VIM is independent of mesh size and becomes relatively small compared to FDM for fine grids, as observed in the CPU time measurements.
5. The estimator requires no additional PDE solves or mesh refinement, making it computationally efficient compared to traditional residual-based or mesh-doubling approaches.

7. Discussion and Limitations

The numerical experiments confirm that the hybrid estimator $\eta(n, h)$ reliably bounds the FDM error and provides a sharp upper bound, particularly for smooth problems.

However, several limitations of the current framework should be acknowledged:

1. **Global Lipschitz Assumption (A3):** The contraction proof for VIM relies on a global Lipschitz condition. For problems with exponential or superlinear growth, this assumption fails, though local Lipschitz conditions may be sufficient if the solution remains within a bounded region.
2. **One-Dimensional Domains:** The compactness proof and the M-matrix structure are specific to 1D. Extending the hybrid estimator to 2D or 3D PDEs requires significant additional work, particularly in establishing the compactness of the inverse operator and the stability of the discrete system.
3. **Dirichlet Boundary Conditions:** The analysis is restricted to homogeneous Dirichlet conditions. Neumann or Robin conditions introduce additional technicalities in the definition of the operator domain and the finite difference stencil.
4. **Computational Cost of VIM:** While VIM is claimed to be cheap, symbolic integration can become expensive for complex nonlinearities. In practice, numerical quadrature can be used to evaluate the integrals, making it feasible for most applications.

Despite these limitations, the framework offers a promising new direction for cheap, reliable error estimation in adaptive numerical methods. A rigorous proof of asymptotic exactness of the estimator $\eta(n, h)$: that the efficiency index $\text{Eff} = \eta(n, h) / \|I_h u_h - u^*\|_\infty$ tends to 1 as $h \rightarrow 0$, remains an open question and is a direction for future investigation. The numerical results in Section 6 suggest this property holds for the problems considered.

8. Conclusion

This paper has introduced a unified operator-theoretic framework for analyzing the Variational Iteration Method and the Finite Difference Method for second-order nonlinear BVPs. The key contributions are:

1. A rigorous proof of the contraction property of VIM under explicit conditions, yielding a computable a priori error bound.
2. A convergence proof for FDM as a finite-rank projection, establishing the $O(h^2)$ rate.
3. A novel hybrid a posteriori error estimator $\eta(n, h)$ that combines the continuous VIM iterate with the discrete FDM solution to provide a reliable and computationally inexpensive bound on the global error.

The numerical experiments validate the theoretical results and demonstrate the efficiency of the estimator. Future work will focus on extending the hybrid estimator to partial differential equations and implementing it in adaptive mesh refinement algorithms.

References

- [1] Ascher, U. M., Mattheij, R. M., & Russell, R. D. (1995). *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. SIAM. <https://epubs.siam.org/doi/book/10.1137/1.9781611971231>
- [2] He, J. H. (1999). Variational iteration method - a kind of non-linear analytical technique. *International Journal of Non-Linear Mechanics*, 34(4), 699-708. [https://doi.org/10.1016/S0020-7462\(98\)00048-1](https://doi.org/10.1016/S0020-7462(98)00048-1)
- [3] He, J. H. (2007). Variational iteration method—some recent results and new interpretations. *Journal of Computational and Applied Mathematics*, 207(1), 3-17. <https://doi.org/10.1016/j.cam.2006.07.009>
- [4] Momani, S., Abuasad, S., & Odibat, Z. (2006). Variational iteration method for solving nonlinear boundary value problems. *Applied Mathematics and Computation*, 183(2), 1351-1358. doi: <http://doi.org/10.1016/j.amc.2006.05.138>
- [5] Noor, M. A., & Mohyud-Din, S. T. (2008). Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*, 11(1), 55-62. <https://doi.org/10.1515/IJNSNS.2008.9.2.141>
- [6] Akpakwa, J. I., Udoaka, O. G., Udofia, E. S., & Edem, I. D. (2026). Inertial averaged type algorithm with norm free dynamic stepsize for split common fixed point problems in Banach spaces. *IORMS Journal of Operational Research*, 4(1), 42-54. <https://journal.iorms.org.ng/index.php/ijorms/article/download/21/24>
- [7] Shobanapriya, M., & Ramachandran, M. (2024). Computing the solution of hybrid fuzzy differential equations with the application of He's variational iteration method. *Journal of Computational Mathematics*, (in press).
- [8] Yadav, S., Singh, M., Singh, S., Heinrich, S. & Kumar, J. (2024). Modified variational iteration method and its convergence analysis for solving nonlinear aggregation population balance equation. *Computers & Fluids*, 106233. <http://doi.org/10.1016/j.compfluid.2024.106233>.
- [9] Matoog, R. T., Abdou, M. A. & Abdel-Aty, M. A. (2023). New algorithms for solving nonlinear mixed integral equations. *AIMS Mathematics*, 8(11), 27488-27512. <https://www.aimspress.com/article/doi/10.3934/math.20231406>
- [10] Scheiber, E. (2016). On the convergence of the variational iteration method. *J. Numer. Anal. Approx. Theory*, 45(1), 87-96. <https://doi.org/10.33993/jnaat451-1053>.
- [11] Agrawal, A. K., & Gupta, Y. (2022). Variational iteration method for solving differential equations in various science and engineering applications: A review. In B. P. Chamola, P. Kumari, & L. Kaur (Eds.), *Emerging advancements in mathematical sciences*. New York. <https://philpapers.org/rec/PRAEAI>
- [12] Edem, I. D., Okon, U. M., & Udoaka, O. G. (2026). Boundary impulse response for the heat equation on a half line: A Laplace transform approach and semigroup interpretation. *IORMS Journal of Operational Research and Management Science*, 4(1), 70-80. <https://journal.iorms.org.ng/index.php/ijorms/article/download/20/26>

-
- [13] Verfürth, R. (1996). *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*. Wiley. <https://search.worldcat.org/title/35042554>
 - [14] Bank, R. E., & Weiser, A. (1982). Some a posteriori error estimators for elliptic partial differential equations. *Mathematics of Computation*, 44(170), 283-301. <https://pubs.ams.org/journals/mcom/1985-44-170/S0025-5718-1985-0777265-X>
 - [15] Evans, L. C. (2010). *Partial Differential Equations*. American Mathematical Society. <https://bookstore.ams.org/gsm-19-r/>
 - [16] Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer. <https://link.springer.com/book/10.1007/978-0-387-70914-7>
 - [17] Titchmarsh, E. C. (1962). *Eigenfunction Expansions Associated with Second-Order Differential Equations, Part I*. Oxford University Press. <https://search.worldcat.org/title/25284469>
 - [18] Edem, I. D., Udoaka, O. G., & Akpan, U. D. (2026). Application of generalized hybrid fixed point theory to nonlinear integral equations in L^p spaces. *IORMS Journal of Operational Research and Management Science*, 4(1), 55-69. <https://journal.iorms.org.ng/index.php/ijorms/article/download/25/25>
 - [19] Mohan, J., & Sood, A. Picard, E. (2023). Picard's method of successive approximation for Fractional Order Initial Value Problem. *Advances in Differential Equations and Control Processes*, 30(4). <http://www.doi.org/10.17654/0974324323019>
 - [20] Edem, I. D., Akpakwa, J., Sampson, M., George, R., Awad, M. M., & Nabwey, H. A. (2026). Strong convergence of the Halpern iteration for monotone α -nonexpansive mappings in uniformly convex ordered Banach spaces. *International Journal of Analysis and Applications*, 24, Article 129. <https://etamaths.com/index.php/ijaa/article/download/4942/1715>
 - [21] Varga, R. S. (2000). *Matrix Iterative Analysis*, Second Edition. Springer. <https://link.springer.com/book/10.1007/978-3-642-05156-2>
 - [22] Edem, I. D., Akpakwa, J., Okon, U. M., Udogworen, W. K., & Udoaka, O. G. (2026). Strictly pseudo-contractive Volterra integral operators in Banach spaces: Weak and strong convergence of the Mann iteration with a numerical analysis extension. *Ktrend - African Journal of Mathematics, Statistics and Computer Science*, 1(2), 1-16. <https://ktrend.org/article.php?id=12>

Copyright and License

Copyright © 2026 The Author(s).

This article is published by *African Journal of Mathematics, Statistics and Computer Science (AJMSCS)* and is distributed under the terms of the Creative Commons Attribution License (CC BY 4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Journal Website: <https://ktrend.org/>