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The Study of Soft Set Theory, Its Algebra and Applications

Udoaka, O. G.

Department of Mathematics, Akwa Ibom State University, Nigeria
Corresponding Author: otobongawasi@aksu.edu.ng

Abstract

Soft set theory is a parameterized framework for the representation of uncertainty in situations where the available information is incomplete, qualitative, or difficult to encode by a probability distribution or membership function. In this paper, I develop a unified treatment of the basic algebra of soft sets and examine how classical algebraic structures can be represented through parameter-dependent approximations. I formulate the principal soft-set operations on compatible parameter domains, establish elementary structural properties, and study soft groups, soft rings, soft semirings, and soft lattices. I further show that, for a fixed parameter set, suitable families of soft subsets inherit commutative idempotent monoid and distributive lattice-type structures under parameterwise union and intersection. To demonstrate the applied value of the framework, I construct a multi-parameter decision model for selecting an alternative from a finite universe. The model combines binary soft information, parameter reduction, and weighted choice values. An illustrative house-selection example shows how the same soft information can be converted into a transparent ranking procedure while preserving the semantic role of each parameter. The analysis emphasizes that the validity of algebraic identities depends on the adopted definitions and parameter domains, and therefore distinguishes fixed-domain operations from extended operations. The paper provides a concise bridge between the foundational theory, algebraic interpretation, and decision-making use of soft sets.

Keywords: soft set theory; soft algebra; soft group; soft ring; soft semiring; soft lattice; uncertainty; decision making; parameter reduction.

1. Introduction

Many mathematical models are designed for exact information, yet practical problems frequently involve attributes that are qualitative, incomplete, context-dependent, or only approximately known. Probability theory, fuzzy sets, rough sets, interval methods, and related approaches provide important tools for uncertainty, but each imposes modelling assumptions that may be unsuitable for some problems. Soft set theory was introduced by Molodtsov as a parameterized alternative in which uncertainty is represented through a family of subsets indexed by descriptive parameters [1]. The framework is attractive because the modeller can work directly with meaningful attributes without first assigning probabilities or grades of membership.

The subsequent development of soft sets produced definitions of soft subset, equality, complement, union, intersection, AND and OR operations, as well as tabular procedures for decision analysis [2, 3]. Later work clarified that the behaviour of these operations depends strongly on the treatment of parameter domains; in particular, identities that appear formally similar to classical set identities can fail when different forms of extended and restricted operations are mixed [4]. This makes the algebraic formulation of soft sets more than a notational exercise: the parameter domain is part of the structure.

Algebraic structures are central to many areas of pure and applied mathematics. Group-theoretic methods appear in lattice-based cryptography [7], character-theoretic studies [8], finitely generated groups [9], and cryptographic key exchange based on nilpotent groups [10]. Lattice and number-theoretic structures likewise occur in quantum-era cryptography and RSA systems [11, 12]. Related studies of fuzzy group actions, automorphisms, and finite semigroup constructions further demonstrate the importance of algebraic structure in parameterized and computational settings [13, 14, 15]. These developments motivate the study of soft versions of groups, rings, semirings, lattices, and related systems.

In this paper, I study soft set theory from both structural and applied viewpoints. The contribution is organized around three questions. First, how should the principal operations of soft sets be formulated so that their algebraic properties are transparent? Second, how can classical algebraic structures be parameterized to obtain soft algebraic systems? Third, how can the resulting parameterized information be used in a finite decision problem? I address these questions using a fixed-domain algebra for the main structural results, then discuss extended operations separately. I also construct a weighted decision procedure and illustrate it with a house-selection problem.

2. Preliminaries

Let U be a nonempty universe, E a set of parameters, and $A \subseteq E$.

Definition 2.1 (Soft set)**.** A soft set over U is a pair (F, A) where

$$F : A \longrightarrow \mathcal{P}(U).$$

For each $e \in A$, the set $F(e)$ is the approximation associated with parameter e [1].

Definition 2.2 (Soft subset and equality)**.** For soft sets (F, A) and (G, B) over U, (F, A) is a soft subset of (G, B), written

$$(F, A)\widetilde{\subseteq}(G, B),$$

if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. They are soft equal when each is a soft subset of the other.

Definition 2.3 (Null and absolute soft sets)**.** For a fixed A, the null soft set Φ_A satisfies $F(e) = \varnothing$ for every $e \in A$, while the absolute soft set $\widetilde{U}_A$ satisfies $F(e) = U$ for every $e \in A$.

Example 2.4. Let

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$$

be six houses and let $A = \{e_1, e_2, e_3\}$ represent the parameters "expensive", "beautiful", and "cheap". Define

$$F(e_1) = \{h_2, h_4\}, \qquad F(e_2) = \{h_1, h_3, h_5\}, \qquad F(e_3) = \{h_1, h_3, h_6\}.$$

Then (F, A) gives a parameterized description of the houses. No numerical membership grade is required.

3. Algebra of Soft Sets

For the main algebraic results in this section, I fix a common parameter set A. This removes ambiguity caused by changing domains and allows the operations to be interpreted pointwise.

3.1. Parameterwise operations

For (F, A) and (G, A) define

$$(F, A)\widetilde{\cup}(G, A) = (H, A), \qquad H(e) = F(e) \cup G(e),$$

and

$$(F, A)\widetilde{\cap}(G, A) = (K, A), \qquad K(e) = F(e) \cap G(e).$$

The complement is

$$(F, A)^c = (F^c, A), \qquad F^c(e) = U \setminus F(e).$$

Proposition 3.1. *On the collection $\mathcal{S}_A(U)$ of all soft sets over U with parameter set A, parameterwise union and intersection are idempotent, commutative, and associative.*

Proof. For every $e \in A$, the approximations are ordinary subsets of U. Hence

$$F(e) \cup F(e) = F(e), \qquad F(e) \cap F(e) = F(e),$$

and ordinary union and intersection are commutative and associative. Equality at every parameter gives the corresponding soft-set identities. □

Proposition 3.2 (Absorption and distributivity)**.** *For $(F, A), (G, A), (H, A) \in \mathcal{S}_A(U)$,*

$$(F, A)\widetilde{\cap}\left((F, A)\widetilde{\cup}(G, A)\right) = (F, A),$$
$$(F, A)\widetilde{\cup}\left((F, A)\widetilde{\cap}(G, A)\right) = (F, A),$$

and each of $\widetilde{\cup}$ and $\widetilde{\cap}$ distributes over the other.

Proof. Each identity follows parameterwise from the corresponding identity in the Boolean algebra $\mathcal{P}(U)$. □

Theorem 3.3. *The algebra*

$$\left(\mathcal{S}_A(U), \widetilde{\cup}, \widetilde{\cap}, \Phi_A, \widetilde{U}_A\right)$$

is a bounded distributive lattice. With parameterwise complement, it is isomorphic to the direct product

$$\prod_{e\in A} \mathcal{P}(U)$$

of Boolean algebras.

Proof. The preceding propositions establish associativity, commutativity, idempotence, absorption, and distributivity. The null and absolute soft sets are respectively the least and greatest elements. The mapping

$$(F, A) \longmapsto (F(e))_{e\in A}$$

is a bijection from $\mathcal{S}_A(U)$ to $\prod_{e\in A} \mathcal{P}(U)$ and preserves union, intersection, and complement. □

This fixed-domain result explains why many familiar laws hold cleanly in soft-set algebra. When parameter sets differ, however, one must specify the operation carefully [3, 4].

3.2. Restricted and extended operations

Let (F, A) and (G, B) have possibly different parameter sets. Their restricted union is defined on $A \cap B$ by

$$H(e) = F(e) \cup G(e),$$

and their restricted intersection by

$$K(e) = F(e) \cap G(e).$$

An extended union on $A \cup B$ may be defined by

$$H(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

The corresponding extended intersection must be stated according to the adopted convention. This distinction is essential because the parameter set is part of a soft set, and therefore two operations with identical symbols need not represent the same algebraic construction when their domains differ [4].

3.3. AND and OR products

For (F, A) and (G, B), define

$$(F, A) \wedge (G, B) = (H, A \times B), \qquad H(a, b) = F(a) \cap G(b),$$

and

$$(F, A) \vee (G, B) = (K, A \times B), \qquad K(a, b) = F(a) \cup G(b).$$

These operations are useful when a pair of parameters is interpreted jointly rather than as a single attribute.

4. Soft Algebraic Structures

The central idea of a soft algebraic structure is that every nonempty approximation carries the corresponding classical algebraic structure.

4.1. Soft groups

Definition 4.1. Let G be a group. A non-null soft set (F, A) over G is a soft group if $F(a)$ is a subgroup of G for every $a \in A$ for which $F(a) \neq \varnothing$ [5].

Example 4.2. Take $G = \mathbb{Z}_4$ under addition modulo 4 and $A = \{a_1, a_2, a_3\}$. Let

$$F(a_1) = \{0\}, \qquad F(a_2) = \{0, 2\}, \qquad F(a_3) = \mathbb{Z}_4.$$

Every approximation is a subgroup, so (F, A) is a soft group.

Proposition 4.3. *If (F, A) and (G, A) are soft groups over a group G, then their parameterwise intersection is a soft group.*

Proof. For each $a \in A$, $F(a)$ and $G(a)$ are subgroups. Their intersection is a subgroup of G. Therefore every nonempty approximation of the soft intersection is a subgroup. □

Remark 4.4. The analogous statement for parameterwise union is generally false because the union of two subgroups need not be a subgroup. This illustrates an important difference between the algebra of arbitrary soft subsets and the algebra of soft substructures.

4.2. Soft rings

Definition 4.5. Let R be a ring. A non-null soft set (F, A) over R is a soft ring if each nonempty $F(a)$ is a subring of R.

Example 4.6. Let $R = \mathbb{Z}$ and $A = \{2, 3, 6\}$. Define

$$F(2) = 2\mathbb{Z}, \qquad F(3) = 3\mathbb{Z}, \qquad F(6) = 6\mathbb{Z}.$$

Each approximation is a subring of $\mathbb{Z}$, hence (F, A) is a soft ring.

Proposition 4.7. *The parameterwise intersection of two soft rings over the same ring and the same parameter set is a soft ring.*

Proof. The intersection of subrings is a subring. Applying this fact for each parameter proves the result. □

4.3. Soft semirings

Definition 4.8. Let $(S, +, \cdot)$ be a semiring. A soft set (F, A) over S is a soft semiring if every nonempty $F(a)$ is a subsemiring of S.

Soft semirings are particularly natural in settings where addition and multiplication model aggregation and composition without additive inverses. Finite semigroup and semiring structures are also relevant to computational and cryptographic constructions [15].

4.4. Soft lattices

Definition 4.9. Let $(L, \vee, \wedge)$ be a lattice. A soft set (F, A) over L is a soft lattice if every nonempty $F(a)$ is a sublattice of L.

Proposition 4.10. *If (F, A) and (G, A) are soft lattices over L, then their parameterwise intersection is a soft lattice.*

Proof. The intersection of two sublattices is closed under $\vee$ and $\wedge$ and is therefore a sublattice whenever nonempty. □

The lattice viewpoint is useful because soft-set information itself possesses a lattice structure under fixed-domain union and intersection, while each approximation may simultaneously be a sublattice of an external lattice. This creates two levels of order: an order among soft sets and an internal order within their approximations.

5. Monoid and Semiring Interpretations

The fixed-domain soft-set algebra admits useful structural interpretations.

Theorem 5.1. *The systems*

$$(\mathcal{S}_A(U), \widetilde{\cup}, \Phi_A) \quad \textit{and} \quad (\mathcal{S}_A(U), \widetilde{\cap}, \widetilde{U}_A)$$

are commutative idempotent monoids.

Proof. Associativity and commutativity follow parameterwise. The identity for $\widetilde{\cup}$ is Φ_A because

$$F(e) \cup \varnothing = F(e),$$

while the identity for $\widetilde{\cap}$ is $\widetilde{U}_A$ because

$$F(e) \cap U = F(e).$$

Idempotence follows from $X \cup X = X$ and $X \cap X = X$. □

Theorem 5.2. *If soft union is interpreted as addition and soft intersection as multiplication, then*

$$\left(\mathcal{S}_A(U), \widetilde{\cup}, \widetilde{\cap}, \Phi_A, \widetilde{U}_A\right)$$

is an idempotent commutative semiring in the lattice-theoretic sense: union is an idempotent commutative monoid operation, intersection is a commutative monoid operation, intersection distributes over union, and Φ_A is absorbing for intersection.

Proof. All statements follow parameterwise from the corresponding identities for subsets of U:

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z), \qquad X \cap \varnothing = \varnothing.$$

□

This representation is useful because it connects soft-set computation to algebraic systems in which idempotent addition is fundamental. It also gives a direct explanation for matrix-style and optimization procedures based on soft information.

6. Decision-Making Model

I now formulate a finite decision model using soft-set data. Let

$$U = \{u_1, u_2, \ldots, u_m\}$$

be the alternatives and

$$A = \{e_1, e_2, \ldots, e_n\}$$

the selected parameters. The soft set (F, A) can be represented by the binary matrix

$$M = (m_{ij}), \qquad m_{ij} = \begin{cases} 1, & u_i \in F(e_j), \\ 0, & u_i \notin F(e_j). \end{cases}$$

6.1. Unweighted choice value

The unweighted choice value of u_i is

$$c_i = \sum_{j=1}^{n} m_{ij}.$$

An alternative with a larger c_i satisfies more of the selected positive parameters.

6.2. Weighted choice value

If the importance of parameter e_j is $w_j \geq 0$, define

$$C_i = \sum_{j=1}^{n} w_j m_{ij}.$$

The preferred alternative is any u_i satisfying

$$C_i = \max_{1 \leq r \leq m} C_r.$$

Weights may be normalized so that $\sum_j w_j = 1$, although normalization does not change the ranking when all weights are multiplied by the same positive constant.

6.3. Parameter reduction

A parameter is redundant for a particular decision table if its removal does not change the relevant decision outcome under the adopted criterion. Parameter reduction should not be confused with rough-set attribute reduction; the two procedures have different objects and preservation conditions [6]. For a practical soft decision model, I recommend checking whether removal of a parameter changes the maximal choice set. If it does not, the parameter may be omitted for that decision instance.

7. Illustrative Application: House Selection

Consider six houses

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$$

and five positive decision parameters:

$$e_1 = \text{beautiful}, \quad e_2 = \text{cheap}, \quad e_3 = \text{modern}, \quad e_4 = \text{spacious}, \quad e_5 = \text{good location}.$$

Suppose the available information gives

$$F(e_1) = \{h_1, h_3, h_5\},$$

$$F(e_2) = \{h_1, h_3, h_6\},$$

$$F(e_3) = \{h_2, h_3, h_4, h_5\},$$

$$F(e_4) = \{h_1, h_4, h_5\},$$

$$F(e_5) = \{h_2, h_3, h_5, h_6\}.$$

The corresponding table is

House	e_1	e_2	e_3	e_4	e_5	c_i
h_1	1	1	0	1	0	3
h_2	0	0	1	0	1	2
h_3	1	1	1	0	1	4
h_4	0	0	1	1	0	2
h_5	1	0	1	1	1	4
h_6	0	1	0	0	1	2

The unweighted procedure produces a tie between h_3 and h_5, each with choice value 4. This is not a defect; it indicates that the binary information alone does not distinguish the two alternatives.

Suppose the decision maker assigns weights

$$(w_1, w_2, w_3, w_4, w_5) = (0.20, 0.30, 0.15, 0.10, 0.25).$$

Then

$$C_1 = 0.20 + 0.30 + 0.10 = 0.60,$$
$$C_2 = 0.15 + 0.25 = 0.40,$$
$$C_3 = 0.20 + 0.30 + 0.15 + 0.25 = 0.90,$$
$$C_4 = 0.15 + 0.10 = 0.25,$$
$$C_5 = 0.20 + 0.15 + 0.10 + 0.25 = 0.70,$$

and

$$C_6 = 0.30 + 0.25 = 0.55.$$

Hence h_3 is the unique preferred alternative under the stated weights.

Proposition 7.1. *If all weights are strictly positive and an alternative u_p satisfies every parameter satisfied by u_q and at least one additional parameter, then $C_p > C_q$.*

Proof. The difference

$$C_p - C_q = \sum_{j=1}^{n} w_j(m_{pj} - m_{qj})$$

contains no negative term under the stated dominance condition and contains at least one strictly positive term. Therefore $C_p - C_q > 0$. □

This simple dominance result shows that the weighted soft-set model respects a natural preference principle: an alternative cannot be ranked below another alternative that it weakly dominates on every positive criterion and strictly dominates on at least one.

8. Discussion

The analysis reveals two complementary roles for soft sets. The first is representational. A soft set records information as a mapping from parameters to subsets, making it suitable for qualitative attributes and incomplete descriptions. The second is structural. Once a parameter domain is fixed, soft sets inherit an algebra from ordinary subsets, and this algebra can be characterized as a bounded distributive lattice and as an idempotent semiring under suitable interpretation.

The distinction between arbitrary soft sets and soft algebraic substructures is important. The intersection of soft groups, rings, or lattices behaves well because the intersection of the corresponding classical substructures is again a substructure. Union does not generally have this property. Thus, closure properties should be established from the underlying algebra rather than assumed from the behaviour of arbitrary subsets.

The decision example also illustrates why soft sets remain useful even when the final ranking is numerical. The numerical stage is secondary: the underlying information is still organized semantically by parameters. The binary matrix makes the information auditable, while weights allow preferences to be introduced explicitly. This separation between descriptive information and preference weights can be valuable in transparent decision-support systems.

The wider algebraic literature reinforces the relevance of structural thinking. Studies involving group theory, character theory, finitely generated groups, nilpotent groups, lattices, number theory, near-rings, automorphisms, and semigroup constructions demonstrate that algebraic structures continue to play important roles in cryptography and mathematical modelling [7, 8, 9, 10, 11, 12, 13, 14, 15]. Soft-set methods offer a way to organize families of such structures when parameters or uncertainty are intrinsic to the problem.

9. Conclusion

In this paper, I investigated soft set theory with emphasis on its algebraic foundations and applications to decision-making problems. By fixing the parameter domain for the principal structural analysis, I established that the collection of soft sets forms a bounded distributive lattice under parameterwise union and intersection. The same structure can also be interpreted as an idempotent commutative semiring, thereby providing an algebraic framework for studying parameterized information and uncertainty.

I further examined soft groups, soft rings, soft semirings, and soft lattices, showing how classical algebraic structures can be represented through parameter-dependent approximations. In particular, the results demonstrate that parameterwise intersection preserves several soft algebraic structures, whereas the corresponding union operation does not necessarily preserve their defining properties. This distinction highlights the importance of the underlying algebraic structure when extending ordinary set operations to soft algebraic systems.

The decision-making model developed in this study demonstrates how soft-set information can be represented using a binary decision matrix and evaluated through both unweighted and weighted choice values. The house-selection example illustrates that parameter weights can distinguish alternatives that are indistinguishable under an unweighted model, while maintaining a transparent relationship between the final decision and the original parameters.

The results establish soft set theory as a useful mathematical framework for combining parameterized uncertainty with algebraic structure and decision analysis. The relationship between soft-set operations, lattice structures, semirings, and classical algebraic systems provides a foundation for further investigations involving soft modules, soft ideals, soft topological structures, fuzzy-soft systems, and computational methods for large parameter spaces.

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